Differentiating the stochastic entropy for compact negatively curved spaces under conformal changes

Francois Ledrappier, Lin Shu

To cite this version:

Francois Ledrappier, Lin Shu. Differentiating the stochastic entropy for compact negatively curved spaces under conformal changes. 2015. hal-00863933v2

HAL Id: hal-00863933

https://hal.science/hal-00863933v2

Preprint submitted on 11 Aug 2015

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DIFFERENTIATING THE STOCHASTIC ENTROPY FOR COMPACT NEGATIVELY CURVED SPACES UNDER CONFORMAL CHANGES

FRANÇOIS LEDRAPPIER AND LIN SHU

Abstract. We consider the universal cover of a closed connected Riemannian manifold of negative sectional curvature. We show that the linear drift and the stochastic entropy are differentiable under any C^3 one-parameter family of C^3 conformal changes of the original metric.

Contents

1. Introduction

Let (M, g) be an m-dimensional closed connected Riemannian manifold, and $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ its universal cover endowed with the lifted Riemannian metric. The fundamental group $G = \pi_1(M)$ acts on $\widetilde{M}$ as isometries such that $M = \widetilde{M}/G$.

We consider the Laplacian $\Delta := \mathrm{Div}\nabla$ on smooth functions on $(\widetilde{M}, \widetilde{g})$ and the corresponding heat kernel function $p(t, x, y), t \in \mathbb{R}_+, x, y \in \widetilde{M}$, which is the fundamental solution

2010 *Mathematics Subject Classification.* 37D40, 58J65.
Key words and phrases. linear drift, locally symmetric space, stochastic entropy.

to the heat equation $\frac{\partial u}{\partial t} = \Delta u$. Denote by Vol the Riemannian volume on $\widetilde{M}$. The following quantities were introduced by Guivarc'h ([**Gu**]) and Kaimanovich ([**K1**]), respectively, and are independent of $x \in \widetilde{M}$:

- the linear drift $\ell := \lim_{t\to+\infty} \frac{1}{t}\int d_{\widetilde{g}}(x,y)p(t,x,y)\ d\mathrm{Vol}(y)$.
- the stochastic entropy $h := \lim_{t\to+\infty} -\frac{1}{t}\int (\ln p(t,x,y))\, p(t,x,y)\ d\mathrm{Vol}(y)$.

Let $\{g^\lambda = e^{2\varphi^\lambda} g:\ |\lambda|<1\}$ be a one-parameter family of conformal changes of $g^0 = g$, where φ^λ's are real valued functions on M such that $(\lambda, x)\mapsto \varphi^\lambda(x)$ is C^3 and $\varphi^0 \equiv 0$. Denote by ℓ_λ, h_λ, respectively, the linear drift and the stochastic entropy for (M, g^λ). We show

Theorem 1.1. *Let (M,g) be a negatively curved closed connected Riemannian manifold. With the above notation, the functions $\lambda\mapsto \ell_\lambda$ and $\lambda \mapsto h_\lambda$ are differentiable at* 0.

For each $\lambda\in(-1,1)$, let Δ^λ be the Laplacian of $(\widetilde{M}, \widetilde{g}^\lambda)$ with heat kernel $p^\lambda(t,x,y), t\in \mathbb{R}_+, x,y\in\widetilde{M}$, and the associated Brownian motion $\omega_t^\lambda, t\geq 0$. The relation between Δ^λ and Δ is easy to be formulated using $g^\lambda = e^{2\varphi^\lambda}g$: for F a C^2 function on $\widetilde{M}$,

$$\Delta^\lambda F = e^{-2\varphi^\lambda}\left(\Delta F + (m-2)\langle \nabla\varphi^\lambda, \nabla F\rangle_g\right) =: e^{-2\varphi^\lambda} L^\lambda F,$$

where we still denote φ^λ its G-invariant extension to $\widetilde{M}$. Let $\widehat{p}^\lambda(t,x,y), t\in\mathbb{R}_+, x,y\in\widetilde{M}$, be the heat kernel of the diffusion process $\widehat{\omega}_t^\lambda, t\geq 0$, corresponding to the operator L^λ in $(\widetilde{M},\widetilde{g})$. We define

- $\widehat{\ell}_\lambda := \lim_{t\to+\infty}\frac{1}{t}\int d_{\widetilde{g}}(x,y)\widehat{p}^\lambda(t,x,y)\ d\mathrm{Vol}(y)$.
- $\widehat{h}_\lambda := \lim_{t\to+\infty} -\frac{1}{t}\int(\ln \widehat{p}^\lambda(t,x,y))\widehat{p}^\lambda(t,x,y)\ d\mathrm{Vol}(y)$.

It is clear that the following hold true providing all the limits exist:

$$\begin{aligned}(d\ell_\lambda/d\lambda)|_{\lambda=0} &= \lim_{\lambda\to 0}\frac{1}{\lambda}(\ell_\lambda - \widehat{\ell}_\lambda) + \lim_{\lambda\to0}\frac{1}{\lambda}(\widehat{\ell}_\lambda - \ell_0) =: (\mathbf{I})_\ell + (\mathbf{II})_\ell,\\ (dh_\lambda/d\lambda)|_{\lambda=0} &= \lim_{\lambda\to 0}\frac{1}{\lambda}(h_\lambda - \widehat{h}_\lambda) + \lim_{\lambda\to0}\frac{1}{\lambda}(\widehat{h}_\lambda - h_0) =: (\mathbf{I})_h + (\mathbf{II})_h.\end{aligned}$$

Here, loosely speaking, $(\mathbf{I})_\ell$ and $(\mathbf{I})_h$ are the infinitesimal drift and entropy affects of simultaneous metric change and time change of the diffusion (when the generator of the diffusion changes from L^λ to Δ^λ), while $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ are the infinitesimal responses to the adding of drifts to ω_t^0 (when the generator of the diffusion changes from Δ to L^λ).

To analyze $(\mathbf{I})_\ell$ and $(\mathbf{I})_h$, we express the above linear drifts and stochastic entropies using the geodesic spray, the Martin kernel and the exit probability of the Brownian motion at infinity. It is known ([**K1**]) that

$$\ell_\lambda = \int_{M_0\times\partial\widetilde{M}} \langle X^\lambda, \nabla^\lambda \ln k_\xi^\lambda\rangle_\lambda\ d\widetilde{\mathbf{m}}^\lambda,\quad h_\lambda = \int_{M_0\times\partial\widetilde{M}} \|\nabla^\lambda\ln k_\xi^\lambda\|_\lambda^2\ d\widetilde{\mathbf{m}}^\lambda, \tag{1.1}$$

where M_0 is a fundamental domain of $\widetilde{M}$, $\partial\widetilde{M}$ is the geometric boundary of $\widetilde{M}$, $X^\lambda(x,\xi)$ is the unit tangent vector of the $\widetilde{g}^\lambda$-geodesic starting from x pointing at ξ, $k_\xi^\lambda(x)$ is the Martin kernel function of ω_t^λ and $\widetilde{\mathbf{m}}^\lambda$ is the harmonic measure associated with Δ^λ. (Exact definitions will appear in Section 2.) Similar formulas also exist for $\widehat{\ell}_\lambda$ and $\widehat{h}_\lambda$ (see Propositions 2.9, 2.16 and (5.13))

$$\widehat{\ell}_\lambda = \int \langle X^0, \nabla^0 \ln k_\xi^\lambda\rangle_0 \; d\widehat{\mathbf{m}}^\lambda,\ \ \widehat{h}_\lambda = \int \|\nabla^0 \ln k_\xi^\lambda(x)\|_0^2 \; d\widehat{\mathbf{m}}^\lambda, \tag{1.2}$$

where $\widehat{\mathbf{m}}^\lambda$ is the harmonic measure related to the operator L^λ. The quantity $(\mathbf{I})_h$ turns out to be zero since the norm and the gradient changes cancel with the measure change, while the Martin kernel function remains the same under time rescaling of the diffusion process (see Section 5, (5.5) and the paragraph after (5.3)). But the metric variation is more involved in $(\mathbf{I})_\ell$ as we can see from the formulas in (1.1) and (1.2) for ℓ_λ and $\widehat{\ell}_\lambda$. In Section 4, using the (g, g^λ)-Morse correspondence maps (see [**Ano1**, **Gro**, **Mor**] and [**FF**]), which are homeomorphisms between the unit tangent bundle spaces in g and g^λ metrics sending g-geodesics to g^λ-geodesics, we are able to identify the differential

$$\left(\overline{X}^\lambda\right)_0'(x,\xi) := \lim_{\lambda\to 0}\frac{1}{\lambda}\left(\overline{X}^\lambda(x,\xi) - \overline{X}^0(x,\xi)\right), \tag{1.3}$$

where now $\overline{X}^\lambda(x,\xi)$ is the horizontal lift of $X^\lambda(x,\xi)$ to $T_{(x,\xi)}S\widetilde{M}$ (see below Section 2.4), using the stable and unstable Jacobi tensors and a family of Jacobi fields arising naturally from the infinitesimal Morse correspondence (Proposition 4.5 and Corollary 4.6). As a consequence, we can express $(\mathbf{I})_\ell$ using k_ξ^0, $\widetilde{\mathbf{m}}^0$ and these terms (see the proof of Theorem 5.1).

If we continue to analyze $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ using (1.1) and (1.2), we have the problem of showing the regularity in λ of the gradient of the Martin kernels. We avoid this by using an idea from Mathieu ([**Ma**]) to study $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ along the diffusion processes. For every point $x \in \widetilde{M}$ and almost every (a.e.) $\widetilde{g}$-Brownian motion path ω^0 starting from x, it is known ([**K1**]) that

$$\lim_{t\to+\infty}\frac{1}{t}d_{\widetilde{g}}(x,\omega_t^0) = \ell_0, \quad \lim_{t\to+\infty} -\frac{1}{t}\ln G(x,\omega_t^0) = h_0, \tag{1.4}$$

where $G(\cdot,\cdot)$ on $\widetilde{M}\times\widetilde{M}$ denotes the Green function for $\widetilde{g}$-Brownian motion. A further study on the convergence of the limits of (1.4) in [**L2**] showed that there are positive numbers σ_0, σ_1 so that the distributions of the variables

$$\mathrm{Z}_{\ell,t}(x) = \frac{1}{\sigma_0\sqrt{t}}\left[d_{\widetilde{g}}(x,\omega_t^0) - t\ell_0\right], \ \ \mathrm{Z}_{h,t}(x) = \frac{1}{\sigma_1\sqrt{t}}\left[-\ln G(x,\omega_t^0) - th_0\right] \tag{1.5}$$

are asymptotically close to the normal distribution as t goes to infinity. Moreover, these limit theorems have some uniformity when we vary the original metric locally in the space of negatively curved metrics. This allows us to identify $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ respectively with the limits

$$\left(\mathbb{E}_\lambda(\frac{1}{\sqrt{t}}d_{\widetilde{g}}(x,\omega_t^0)) - \mathbb{E}_0(\frac{1}{\sqrt{t}}d_{\widetilde{g}}(x,\omega_t^0))\right), \quad -\left(\mathbb{E}_\lambda(\frac{1}{\sqrt{t}}\ln G(x,\omega_t^0)) - \mathbb{E}_0(\frac{1}{\sqrt{t}}\ln G(x,\omega_t^0))\right),$$

where we take $\lambda = \pm 1/\sqrt{t}$ and $\mathbb{E}_\lambda$ is the expectation with respect to the transition probability of the L^λ process. (More details of the underlying idea will be exposed in Section 3.1 after we introduce the corresponding notations.) Note that all $\widehat{\omega}_t^\lambda$ starting from x can be simultaneously represented as random processes on the probability space $(\Theta, \mathbb{Q})$ of a standard m-dimensional Euclidean Brownian motion. By using the Girsanov-Cameron-Martin formula on manifolds (cf. [**El**]), we are able to compare $\mathbb{E}_\lambda$ with $\mathbb{E}_0$ on the same probability space of continuous path spaces. As a consequence, we show

$$\textbf{(II)}_\ell = \lim_{t\to+\infty} \mathbb{E}_0(\mathrm{Z}_{\ell,t}\mathrm{M}_t) \ \text{ and } \ \textbf{(II)}_h = \lim_{t\to+\infty} \mathbb{E}_0(\mathrm{Z}_{h,t}\mathrm{M}_t),$$

where each M_t is a random process on $(\Theta, \mathbb{Q})$ recording the change of metrics along the trajectories of Brownian motion to be specified in Section 5. We will further specify $\textbf{(II)}_\ell$ and $\textbf{(II)}_h$ in Theorem 5.1 using properties of martingales and the Central Limit Theorems for the linear drift and the stochastic entropy.

An immediate consequence of Theorem 1.1 is that $D_\lambda := h_\lambda/\ell_\lambda$, which is proportional [1] to the Hausdorff dimension of the distribution of the Brownian motion ω^λ at the infinity boundary of $\widetilde{M}$ ([**L1**]), is also differentiable in λ. Let $\Re(M)$ be the manifold of negatively curved C^3 metrics on M. Another consequence of Theorem 1.1 is that

Theorem 1.2. *Let (M,g) be a negatively curved compact connected Riemannian manifold. If it is locally symmetric, then for any C^3 curve $\lambda \in (-1,1) \mapsto g^\lambda \in \Re(M)$ of conformal changes of the metric $g^0 = g$ with constant volume,*

$$(dh_\lambda/d\lambda)|_{\lambda=0} = 0, \quad (d\ell_\lambda/d\lambda)|_{\lambda=0} = 0.$$

In case M is a Riemannian surface, the stochastic entropy remains the same for $g \in \Re(M)$ with constant volume. This is because any $g \in \Re(M)$ is a conformal change of a metric with constant curvature by the Uniformization Theorem, metrics with the same constant curvature have the same stochastic entropy by (1.1) and the constant curvature depends only on the volume by the Gauss-Bonnet formula. Indeed, our formula (Theorem 5.1, (5.2)) yields $dh_\lambda/d\lambda \equiv 0$ in the case of surfaces if the volume is constant.

When M has dimension at least 3, it is interesting to know whether the converse direction of Theorem 1.2 for the stochastic entropy holds. We have the following question.

Let (M,g) be a negatively curved compact connected Riemannian manifold with dimension greater than 3. Do we have that (M,g) is locally symmetric if and only if for any C^3 curve $\lambda \in (-1,1) \mapsto g^\lambda \in \Re(M)$ of constant volume with $g^0 = g$, the mapping $\lambda \mapsto h_\lambda$ is differentiable and has a critical point at 0?

[1] D_λ is $\frac{1}{\iota}$ the Hausdorff dimension of the exit measure for the ι-Busemann distance (cf. Section 3.1).

We will present the proof of Theorem 1.1 and the above discussion in a more general setting. Indeed, whereas the statements so far deal only with the Brownian motion on $\widetilde{M}$, proofs of the limit theorems such as (1.4) or (1.5) involve the laminated Brownian motion associated with the stable foliation of the geodesic flow on the unit tangent bundle $\varpi : SM \to M$. As recalled in Section 2.1, the *stable foliation* $\mathcal{W}$ of the geodesic flow is a Hölder continuous lamination, the leaves of which are locally identified with $\widetilde{M}$. A differential operator $\mathcal{L}$ on (the smooth functions on) SM with continuous coefficients and $\mathcal{L}1 = 0$ is said to be *subordinate to the stable foliation* $\mathcal{W}$, if for every smooth function F on SM the value of $\mathcal{L}(F)$ at $v \in SM$ only depends on the restriction of F to $W^s(v)$. We are led to consider the family $\mathcal{L}^\lambda$ of subordinated operators to the stable foliation, given, for F smooth on SM, by

$$\mathcal{L}^\lambda F \ = \ \Delta F + (m-2)\langle \nabla(\varphi^\lambda \circ \varpi), \nabla F\rangle,$$

where Laplacian, gradient and scalar product are taken along the leaves of the lamination and for the metric lifted from the metric $\widetilde{g}$ on $\widetilde{M}$. Diffusions associated to a general subordinated operator of the form $\Delta + Y$, where Y is a laminated vector field, have been studied by Hamenstädt ([**H2**]). We recall her results and several tools in Section 2. In particular, the diffusions associated to $\mathcal{L}^\lambda$ have a drift $\overline{\ell}_\lambda$ and an entropy $\overline{h}_\lambda$ that coincide with respectively $\widehat{\ell}_\lambda$ and $\widehat{h}_\lambda$. Convergences (1.4) and (1.5) are now natural in this framework. Then, our strategy is to construct all the laminated diffusions associated to the different λ and starting from the same point on the same probability space and to compute the necessary limits as expectations of quantities on that probability space that are controlled by probabilistic arguments. For each $\mathbf{v} \in S\widetilde{M}$, the stable manifold $W^s(\mathbf{v})$ is identified with $\widetilde{M}$ (or a $\mathbb{Z}$ quotient of $\widetilde{M}$).[2] As recalled in Subsection 2.5, the diffusions are constructed on $\widetilde{M}$ as projections of solutions of stochastic differential equations on the orthogonal frame bundle $O(\widetilde{M})$ with the property that only the drift part depends on λ (and on $\mathbf{v}$). The quantities $\overline{\ell}_\lambda$ and $\overline{h}_\lambda$ can be read now on the directing probability space, so that we can compute $(\mathbf{II})_\ell$ and $(\mathbf{II})_h$ in Section 4. We cannot do this computation in such a direct manner for a general perturbation $\lambda \mapsto g^\lambda \in \Re(M)$ and this is the reason why we restrict our analysis in this paper to the case of conformal change. But the idea of analyzing the linear drift and the stochastic entropy using the stochastic differential equations can be further polished to treat the general case ([**LS2**]).

We thus will obtain explicit formulas for $(d\ell_\lambda/d\lambda)|_{\lambda=0}$ and $(dh_\lambda/d\lambda)|_{\lambda=0}$ in Theorem 5.1, which, in particular, will imply Theorem 1.1. Finally, Theorem 1.2 can be deduced either using the formulas in Theorem 5.1 or merely using Theorem 1.1 and the existing results concerning the regularity of volume entropy for compact negatively curved spaces under conformal changes from [**Ka**, **KKPW**].

We will arrange the paper as follow. Section 2 is to introduce the linear drift and stochastic entropy for a laminated diffusion of the unit tangent bundle with generator

[2]When $\mathbf{v}$ is on a periodic orbit, then $W^s(\mathbf{v})$ is a cylinder identified with the quotient of $\widetilde{M}$ by the action of one element of G represented by the closed geodesic.

$\Delta + Y$ ([**H2**]) and to understand them by formulas using pathwise limits and integral formulas at the boundary, respectively. There are two key auxiliary properties for the computations of the differentials of $\widehat{\ell}^\lambda, \widehat{h}^\lambda$ in λ: one is the Central Limit Theorems for the linear drift and the stochastic entropy; the other is the probabilistic pathwise relations between the distributions of the diffusions of different generators. They will be addressed in Subsections 2.5 and 2.6, respectively. In Section 3, we will compute separately the differentials of the linear drift and the stochastic entropy associated to a one-parameter of laminated diffusions with generators $\Delta + Y + Z^\lambda$. Section 4 is to use the infinitesimal Morse correspondence ([**FF**]) to derive the differential $\lambda \mapsto \overline{X}^\lambda$ for any general C^3 curve $\lambda \mapsto g^\lambda$ contained in $\Re(M)$. The last section is devoted to the proof of Theorem 1.1 and Theorem 1.2 as was mentioned in the previous paragraph.

2. Foliated diffusions

In this section, we recall results from the literature and we fix notations about the stable foliation in negative sectional curvature, the properties of the diffusions subordinated to the stable foliation and the construction of these diffusions as solutions of SDE.

2.1. **Harmonic measures for the stable foliation.** Recall that $(\widetilde{M}, \widetilde{g})$ is the universal cover space of (M, g), a negatively curved m-dimensional closed connected Riemannian manifold with fundamental group G. Two geodesics in $\widetilde{M}$ are said to be equivalent if they remain a bounded distance apart and the space of equivalent classes of unit speed geodesics is the geometric boundary $\partial\widetilde{M}$. For each $(x, \xi) \in \widetilde{M} \times \partial\widetilde{M}$, there is a unique unit speed geodesic $\gamma_{x,\xi}$ starting from x belonging to $[\xi]$, the equivalent class of ξ. The mapping $\xi \mapsto \dot{\gamma}_{x,\xi}(0)$ is a homeomorphism π_x^{-1} between $\partial\widetilde{M}$ and the unit sphere $S_x\widetilde{M}$ in the tangent space at x to $\widetilde{M}$. So we will identify $S\widetilde{M}$, the unit tangent bundle of $\widetilde{M}$, with $\widetilde{M} \times \partial\widetilde{M}$.

Consider the geodesic flow $\mathbf{\Phi}_t$ on $S\widetilde{M}$. For each $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, its *stable manifold* with respect to $\mathbf{\Phi}_t$, denoted $W^s(\mathbf{v})$, is the collection of initial vectors $\mathbf{w}$ of geodesics $\gamma_{\mathbf{w}} \in [\xi]$ and can be identified with $\widetilde{M} \times \{\xi\}$. Extend the action of G continuously to $\partial\widetilde{M}$. Then SM, the unit tangent bundle of M, can be identified with the quotient of $\widetilde{M} \times \partial\widetilde{M}$ under the diagonal action of G. Clearly, for $\psi \in G$, $\psi(W^s(\mathbf{v})) = W^s(D\psi(\mathbf{v}))$ so that the collection of $W^s(\mathbf{v})$ defines a lamination $\mathcal{W}$ on SM, the so-called *stable foliation* of SM. The leaves of the stable foliation $\mathcal{W}$ are discrete quotients of $\widetilde{M}$, which are naturally endowed with the Riemannian metric induced from $\widetilde{g}$. For $v \in SM$, let $W^s(v)$ be the leaf of $\mathcal{W}$ containing v. Then $W^s(v)$ is a C^2 immersed submanifold of SM depending continuously on v in the C^2-topology ([**SFL**]). (More properties of the stable foliation and of the geodesic flow will appear in Section 2.4.)

Let $\mathcal{L}$ be an operator on (the smooth functions on) SM with continuous coefficients which is subordinate to the stable foliation $\mathcal{W}$. A Borel measure $\mathbf{m}$ on SM is called

$\mathcal{L}$*-harmonic* if it satisfies

$$\int \mathcal{L}(f)\ d\mathbf{m} = 0$$

for every smooth function f on SM. If the restriction of $\mathcal{L}$ to each leaf is elliptic, it is true by [**Ga**] that there always exist harmonic measures and the set of harmonic probability measures is a non-empty weak* compact convex set of measures on SM. A harmonic probability measure $\mathbf{m}$ is *ergodic* if it is extremal among harmonic probability measures.

In this paper, we are interested in the case $\mathcal{L} = \Delta + Y$, where Δ is the laminated Laplacian and Y is a section of the tangent bundle of $\mathcal{W}$ over SM of class $C_s^{k,\alpha}$ for some $k \geq 1$ and $\alpha \in [0,1)$ in the sense that Y and its leafwise jets up to order k along the leaves of $\mathcal{W}$ are Hölder continuous with exponent α ([**H2**]). Let $\mathbf{m}$ be an $\mathcal{L}$-harmonic measure. We can characterize it by describing its lift on $S\widetilde{M}$.

Extend $\mathcal{L}$ to a G-equivariant operator on $S\widetilde{M} = \widetilde{M} \times \partial\widetilde{M}$ which we shall denote with the same symbol. It defines a Markovian family of probabilities on $\widetilde{\Omega}_+$, the space of paths of $\widetilde{\omega} : [0, +\infty) \to S\widetilde{M}$, equipped with the smallest σ-algebra $\mathcal{A}$ for which the projections $R_t :\ \widetilde{\omega} \mapsto \widetilde{\omega}(t)$ are measurable. Indeed, for $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, let $\mathcal{L}_{\mathbf{v}}$ denote the laminated operator of $\mathcal{L}$ on $W^s(\mathbf{v})$. It can be regarded as an operator on $\widetilde{M}$ with corresponding heat kernel functions $p_{\mathbf{v}}(t, y, z)$, $t \in \mathbb{R}_+, y, z \in \widetilde{M}$. Define

$$\mathbf{p}(t, (x, \xi), d(y, \eta)) = p_{\mathbf{v}}(t, x, y) d\mathrm{Vol}(y)\delta_\xi(\eta),$$

where $\delta_\xi(\cdot)$ is the Dirac function at ξ. Then the diffusion process on $W^s(\mathbf{v})$ with infinitesimal operator $\mathcal{L}_{\mathbf{v}}$ is given by a Markovian family $\{\mathbb{P}_{\mathbf{w}}\}_{\mathbf{w} \in \widetilde{M} \times \{\xi\}}$, where for every $t > 0$ and every Borel set $A \subset \widetilde{M} \times \partial\widetilde{M}$ we have

$$\mathbb{P}_{\mathbf{w}}\left(\{\widetilde{\omega} :\ \widetilde{\omega}(t) \in A\}\right) = \int_A \mathbf{p}(t, \mathbf{w}, d(y, \eta)).$$

The following concerning $\mathcal{L}$-harmonic measures holds true.

Proposition 2.1. *(*[**Ga**, **H2**]*) Let $\widetilde{\mathbf{m}}$ be the G-invariant measure which extends an $\mathcal{L}$-harmonic measure $\mathbf{m}$ on $\widetilde{M} \times \partial\widetilde{M}$. Then*

i) *the measure $\widetilde{\mathbf{m}}$ satisfies, for all $f \in C_c^2(\widetilde{M} \times \partial\widetilde{M})$,*

$$\int_{\widetilde{M} \times \partial\widetilde{M}} \left(\int_{\widetilde{M} \times \partial\widetilde{M}} f(y, \eta)\mathbf{p}(t, (x, \xi), d(y, \eta)) \right) d\widetilde{\mathbf{m}}(x, \xi) = \int_{\widetilde{M} \times \partial\widetilde{M}} f(x, \xi)\ d\widetilde{\mathbf{m}}(x, \xi);$$

ii) *the measure $\widetilde{\mathbb{P}} = \int \mathbb{P}_{\mathbf{v}}\ d\widetilde{\mathbf{m}}(\mathbf{v})$ on $\widetilde{\Omega}_+$ is invariant under the shift map $\{\sigma_t\}_{t \in \mathbb{R}_+}$ on $\widetilde{\Omega}_+$, where $\sigma_t(\widetilde{\omega}(s)) = \widetilde{\omega}(s + t)$ for $s > 0$ and $\widetilde{\omega} \in \widetilde{\Omega}_+$;*

iii) *the measure $\widetilde{\mathbf{m}}$ can be expressed locally at $\mathbf{v} = (x, \xi) \in S\widetilde{M}$ as $d\widetilde{\mathbf{m}} = k(y, \eta)(dy \times d\nu(\eta))$, where ν is a finite measure on $\partial\widetilde{M}$ and, for ν-almost every η, $k(y, \eta)$ is a positive function on $\widetilde{M}$ which satisfies $\Delta(k(y, \eta)) - \mathrm{Div}(k(y, \eta)Y(y, \eta)) = 0$.*

The group G acts naturally and discretely on the space $\widetilde{\Omega}_+$ of continuous paths in $S\widetilde{M}$ with quotient the space Ω_+ of continuous paths in SM, and this action commutes with the shift $\sigma_t, t \geq 0$. Therefore, the measure $\widetilde{\mathbb{P}}$ is the extension of a finite, shift invariant measure $\mathbb{P}$ on Ω_+. Note that SM can be identified with $M_0 \times \partial\widetilde{M}$, where M_0 is a fundamental domain of $\widetilde{M}$. Hence we can also identify Ω_+ with the lift of its elements in $\widetilde{\Omega}_+$ starting from M_0. Elements in Ω_+ will be denoted by ω. We will also clarify the notions whenever there is an ambiguity. In all the paper, we will normalize the harmonic measure $\mathbf{m}$ to be a probability measure, so that the measure $\mathbb{P}$ is also a probability measure. We denote by $\mathbb{E}_{\mathbb{P}}$ the corresponding expectation symbol.

Call $\mathcal{L}$ *weakly coercive*, if $\mathcal{L}_{\mathbf{v}}$, $\mathbf{v} \in S\widetilde{M}$, are weakly coercive in the sense that there are a number $\varepsilon > 0$ (independent of $\mathbf{v}$) and, for each $\mathbf{v}$, a positive $(\mathcal{L}_{\mathbf{v}}+\varepsilon)$-superharmonic function F on $\widetilde{M}$ (i.e. $(\mathcal{L}_{\mathbf{v}} + \varepsilon)F \geq 0$). For instance, if $Y \equiv 0$, then $\mathcal{L} = \Delta$ is weakly coercive and it has a unique $\mathcal{L}$-harmonic measure $\mathbf{m}$, whose lift in $S\widetilde{M}$ satisfies $d\widetilde{\mathbf{m}} = dx \times d\widetilde{\mathbf{m}}_x$, where dx is proportional to the volume element and $\widetilde{\mathbf{m}}_x$ is the hitting probability at $\partial\widetilde{M}$ of the Brownian motion starting at x. Consequently, in this case, the function k in Proposition 2.1 is the Martin kernel function. This relation is not clear for general weakly coercive $\mathcal{L}$.

A nice property for the laminated diffusion associated with a weakly coercive operator is that the semi-group $\sigma_t, t \geq 0$, of transformations of Ω_+ has strong ergodic properties with respect to the probability $\mathbb{P}$, provided Y satisfies some mild condition. Recall that a measure preserving semi flow $\sigma_t, t \geq 0$, of transformations of a probability space $(\Omega, \mathbb{P})$ is called mixing if for any bounded measurable functions F_1, F_2 on Ω,

$$\lim_{t\to+\infty} \mathbb{E}_{\mathbb{P}}(F_1\,(F_2 \circ \sigma_t)) \;=\; \mathbb{E}_{\mathbb{P}}(F_1)\mathbb{E}_{\mathbb{P}}(F_2).$$

Proposition 2.2. *Let $\mathcal{L} = \Delta + Y$ be subordinated to the stable foliation and such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely. Assume that $\mathcal{L}$ is weakly coercive. Let $\mathbf{m}$ be the unique invariant measure, $\mathbb{P}$ the associated probability measure on Ω_+. The shift semi-flow $\sigma_t, t \geq 0$, is mixing on $(\Omega_+, \mathbb{P})$.*

(Note that Y is a section of the tangent bundle of $\mathcal{W}$ over SM of class $C^{k,\alpha}_s$ and that Y^* is a section of the cotangent bundle of $\mathcal{W}$ over SM of class $C^{k,\alpha}_s$, the duality being defined by the metric inherited from $\widetilde{M}$. The hypothesis is that this 1-form, seen as a 1-form on $\widetilde{M}$, is closed.)

Proof. The classical proof that a weakly coercive subordinated operator admits a unique harmonic measure (see [**Ga**], [**L2**], [**Y**] for the case of Δ) shows in fact the mixing property if F_1 and F_2 are functions on Ω_+ that depends only on the starting point of the path and are continuous as functions on SM. The mixing property is extended first to bounded measurable functions on Ω_+ that depends only on the starting point of the path by (L^2, say) density, then to functions depending on a finite number of coordinates in the space of paths by the Markov property and finally to all bounded measurable functions by L^2 density. $\square$

2.2. **Linear drift and stochastic entropy for laminated diffusion.** Let $\mathbf{m}$ be an $\mathcal{L}$-harmonic measure and $\widetilde{\mathbf{m}}$ be its G-invariant extension in $S\widetilde{M}$. Choose a fundamental domain M_0 of $\widetilde{M}$ and identify SM with $M_0 \times \partial\widetilde{M}$. We normalize $\widetilde{\mathbf{m}}$ so that $\widetilde{\mathbf{m}}(M_0 \times \partial\widetilde{M}) = 1$ (so that the measure $\mathbb{P}$ is a probability; we denote $\mathbb{E}_{\mathbb{P}}$ the corresponding expectation). Let $d_{\mathcal{W}}$ denote the leafwise metric on the stable foliation of $S\widetilde{M}$. Then it can be identified with $d_{\widetilde{g}}$ on $\widetilde{M}$ on each leaf. We define

$$\ell_{\mathcal{L}}(\mathbf{m}) := \lim_{t\to+\infty} \frac{1}{t} \int_{M_0\times\partial\widetilde{M}} d_{\mathcal{W}}((x,\xi),(y,\eta))\mathbf{p}(t,(x,\xi),d(y,\eta))\ d\widetilde{\mathbf{m}}(x,\xi),$$

$$h_{\mathcal{L}}(\mathbf{m}) := \lim_{t\to+\infty} -\frac{1}{t} \int_{M_0\times\partial\widetilde{M}} (\ln \mathbf{p}(t,(x,\xi),(y,\eta)))\ \mathbf{p}(t,(x,\xi),d(y,\eta))\ d\widetilde{\mathbf{m}}(x,\xi).$$

Equivalently, by using $\widetilde{\mathbb{P}}$ in Proposition 2.1, we see that

$$\ell_{\mathcal{L}}(\mathbf{m}) = \lim_{t\to+\infty} \frac{1}{t} \int_{\omega(0)\in M_0\times\partial\widetilde{M}} d_{\mathcal{W}}(\omega(0),\omega(t))\ d\widetilde{\mathbb{P}}(\omega) = \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}_{\mathbb{P}}(d_{\mathcal{W}}(\omega(0),\omega(t))),$$

$$h_{\mathcal{L}}(\mathbf{m}) = \lim_{t\to+\infty} -\frac{1}{t} \int_{\omega(0)\in M_0\times\partial\widetilde{M}} \ln \mathbf{p}(t,\omega(0),\omega(t))\ d\widetilde{\mathbb{P}}(\omega) = \lim_{t\to+\infty} -\frac{1}{t}\mathbb{E}_{\mathbb{P}}(\ln \mathbf{p}(t,\omega(0),\omega(t))).$$

Call $\ell_{\mathcal{L}}(\mathbf{m})$ the *linear drift of $\mathcal{L}$ for* $\mathbf{m}$, and $h_{\mathcal{L}}(\mathbf{m})$ the *(stochastic) entropy of $\mathcal{L}$ for* $\mathbf{m}$. In case there is a unique $\mathcal{L}$-harmonic measure $\mathbf{m}$, we will write $\ell_{\mathcal{L}} := \ell_{\mathcal{L}}(\mathbf{m})$ and $h_{\mathcal{L}} := h_{\mathcal{L}}(\mathbf{m})$ and call them the *linear drift* and the *(stochastic) entropy for $\mathcal{L}$*, respectively.

Clearly, $h_{\mathcal{L}}(\mathbf{m})$ is nonnegative by definition. We are interested in the case that $h_{\mathcal{L}}(\mathbf{m})$ is positive. When $\mathcal{L} = \Delta$, this is true ([**K1**, Theorem 10]). In general, there exist weakly coercive $\mathcal{L}$'s which admit uncountably many harmonic measures with zero entropy ([**H2**]).

Let $\mathcal{L}$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely. For $v \in SM$, let $\overline{X}(v)$ be the tangent vector to $W^s(v)$ that projects on v and let

$$\mathrm{pr}(-\langle \overline{X}, Y\rangle) := \sup\left\{h_\mu - \int \langle \overline{X}, Y\rangle\ d\mu : \quad \mu \in \mathcal{M}\right\}$$

be the pressure of the function $-\langle \overline{X}, Y\rangle$ on SM with respect to the geodesic flow Φ_t, where $\mathcal{M}$ is the set of Φ_t-invariant probability measures on SM and h_μ is the entropy of μ with respect to Φ_t. Then,

Proposition 2.3 ([**H2**]). *Let $\mathcal{L} = \Delta + Y$ be subordinated to the stable foliation and such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely. Then, $h_{\mathcal{L}}(\mathbf{m})$ is positive if and only if pr$(-\langle \overline{X}, Y\rangle)$ is positive, and each one of the two positivity properties implies that $\mathcal{L}$ is weakly coercive,* $\mathbf{m}$ *is the unique $\mathcal{L}$-harmonic measure and $\ell_{\mathcal{L}}(\mathbf{m})$ is positive.*

In particular, when we consider $\Delta + Z^\lambda$, where $Z^\lambda := (m-2)\nabla(\varphi^\lambda \circ \varpi)$ and φ^λ's are real valued functions on M such that $(\lambda, x) \mapsto \varphi^\lambda(x)$ is C^3 and $\varphi^0 \equiv 0$, the pressure of $-\langle Z^\lambda, \overline{X}\rangle$ is positive for λ close enough to 0.

2.3. **Linear drift and stochastic entropy for laminated diffusions: pathwise limits.** By ergodicity of the shift semi-flow, it is possible to evaluate the linear drift and stochastic entropy along typical paths. Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and $\mathrm{pr}(-\langle \overline{X}, Y\rangle) > 0$. Let $\mathbf{m}$ be the unique $\mathcal{L}$-harmonic measure. By Proposition 2.2 the measure $\mathbb{P}$ associated to $\mathbf{m}$ is ergodic for the shift flow on Ω_+. The following well known fact follows then from Kingman's Subadditive Ergodic Theorem ([**Ki**]). For $\mathbb{P}$-almost all paths $\omega \in \Omega_+$, we still denote by ω its lift in $\widetilde{\Omega}_+$ with $\omega(0) \in M_0$ and we have

$$\lim_{t\to+\infty} \frac{1}{t} d_{\mathcal{W}}(\omega(0), \omega(t)) = \ell_{\mathcal{L}}. \tag{2.1}$$

Similarly, we can characterize $h_{\mathcal{L}}$ using the Green function along the trajectories. For each $\mathbf{v} = (x, \xi) \in \widetilde{M} \times \partial \widetilde{M}$, we can regard $\mathcal{L}_{\mathbf{v}}$ as an operator on $\widetilde{M}$. Since it is weakly coercive, there exists the corresponding Green function $G_{\mathbf{v}}(\cdot, \cdot)$ on $\widetilde{M} \times \widetilde{M}$, defined for $x \neq y$ by

$$G_{\mathbf{v}}(x, y) := \int_0^\infty p_{\mathbf{v}}(t, x, y)\, dt.$$

Define the *Green function* $\mathbf{G}(\cdot, \cdot)$ *on* $S\widetilde{M} \times S\widetilde{M}$ as being

$$\mathbf{G}((y, \eta), (z, \zeta)) := G_{(y,\eta)}(y, z)\delta_\eta(\zeta), \quad \text{for } (y, \eta), (z, \zeta) \in S\widetilde{M},$$

where $\delta_\eta(\cdot)$ is the Dirac function at η. We have the following proposition concerning $h_{\mathcal{L}}$.

Proposition 2.4. *Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and $pr(-\langle \overline{X}, Y\rangle) > 0$. Then for $\mathbb{P}$-a.e. paths $\omega \in \Omega_+$, we have*

$$h_{\mathcal{L}} = \lim_{t\to+\infty} -\frac{1}{t} \ln \mathbf{p}(t, \omega(0), \omega(t)) \tag{2.2}$$

$$= \lim_{t\to+\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)). \tag{2.3}$$

Contrarily to the distance, the function $-\ln \mathbf{p}$ is not elementarily subadditive along the trajectories and the argument used to establish (2.1) has to be modified. We will use the trick of [**L3**] to show that there exists a convex function $h_{\mathcal{L}}(s)$, $s > 0$, such that for $\mathbb{P}$-a.e. paths $\omega \in \Omega_+$, for any $s > 0$,

$$h_{\mathcal{L}}(s) = \lim_{t\to+\infty} -\frac{1}{t} \ln \mathbf{p}(st, \omega(0), \omega(t)). \tag{2.4}$$

Setting $s = 1$ in (2.4) gives that $\lim_{t\to+\infty} -\frac{1}{t} \ln \mathbf{p}(t, \omega(0), \omega(t))$ exists and is $h_{\mathcal{L}}(1)$. Moreover, $h_{\mathcal{L}}(1) \leq h_{\mathcal{L}}$ by Fatou's Lemma. Then, (2.3) and (2.2) will follow once we show that for $\mathbb{P}$-almost all paths $\omega \in \Omega_+$,

$$\lim_{t\to+\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)) = \inf_{s>0}\{h_{\mathcal{L}}(s)\} \geq h_{\mathcal{L}}. \tag{2.5}$$

To show (2.4) and (2.5), we need some detailed descriptions of $p_{\mathbf{v}}(t,x,y)$. First, we have a variant of Moser's parabolic Harnack inequality ([**Mos**]) (see [**St**, **T**] and also [**Sa**]).

Lemma 2.5. *There exist $A, \varsigma > 0$ such that for any $\mathbf{v} \in S\widetilde{M}$, $t \geq 1$, $\frac{1}{2} \leq t' \leq 1$, $x, x', y, y' \in \widetilde{M}$ with $d(x,x') \leq \varsigma$, $d(y,y') \leq \varsigma$,*

$$p_{\mathbf{v}}(t,x,y) \geq A p_{\mathbf{v}}(t-t', x', y'). \tag{2.6}$$

Next, we have the exponential decay property of $p_{\mathbf{v}}(t,x,y)$ in time t.

Lemma 2.6. *([**H2**, p.76]) There exist $B, \varepsilon > 0$ independent of $\mathbf{v}$ such that*

$$p_{\mathbf{v}}(t,x,y) \leq B \cdot e^{-\varepsilon t}, \ \text{for all } y \in \widetilde{M} \text{ and } t \geq 1. \tag{2.7}$$

A Gaussian like upper bound for $p_{\mathbf{v}}(t,x,y)$ is also valid.

Lemma 2.7. *([**Sa**, Theorem 6.1]) There exist constants C_1, C_2, K_1 such that for any $\mathbf{v} \in S\widetilde{M}$, $t > 0$ and $x, y \in \widetilde{M}$, we have*

$$p_{\mathbf{v}}(t,x,y) \leq \frac{1}{\mathrm{Vol(x, \sqrt{t})Vol(y, \sqrt{t})}} \exp\left[C_1(1 + bt + \sqrt{K_1 t}) - \frac{d^2(x,y)}{C_2 t}\right].$$

Let $b > 0$ be an upper bound of $\|Y\|$. We have the following lower bound for $p_{\mathbf{v}}(t,x,y)$.

Lemma 2.8. *([**W**, Theorem 3.1]) Let $\beta = \sqrt{K}(m-1) + b$, where $K \geq 0$ is such that Ricci $\geq -K(m-1)$. Then for any $\mathbf{v} \in S\widetilde{M}$, $t, \sigma > 0$ and $x, y \in \widetilde{M}$, we have*

$$p_{\mathbf{v}}(t,x,y) \geq (4\pi t)^{-\frac{m}{2}} \exp\left[-(\frac{1}{4t} + \frac{\sigma}{3\sqrt{2t}}) d^2(x,y) - \frac{\beta^2 t}{4} - \left(\frac{\beta^2}{4\sigma} + \frac{2m\sigma}{3}\right)\sqrt{2t}\right]. \tag{2.8}$$

Proof of Proposition 2.4. We first show (2.4). Given $s > 0$, for $\omega \in \Omega_+$, define

$$F(s, \omega, t) := -\ln(\mathbf{p}(st-1, \omega(0), \omega(t)) \cdot \widetilde{A}),$$

where $\widetilde{A} = A^2 \inf_{z \in \widetilde{M}} \mathrm{Vol}(B(z, \varsigma))$ and A, ς are as in Lemma 2.5. Then for $t, t' \geq 1/s$, $\omega \in \Omega_+$,

$$F(s, \omega, t+t') \leq F(s, \omega, t) + F(s, \sigma_t(\omega), t').$$

This follows by the semi-group property of $\mathbf{p}$ and (2.6) since

$$\begin{aligned}
\mathbf{p}\left(s(t+t') - 1, \omega(0), \omega(t+t')\right) &= \int \mathbf{p}(st - \frac{1}{2}, \omega(0), z) \mathbf{p}(st' - \frac{1}{2}, z, \omega(t+t'))\ dz \\
&\geq \int_{B(\omega(t), \varsigma)} \mathbf{p}(st - \frac{1}{2}, \omega(0), z) \mathbf{p}(st' - \frac{1}{2}, z, \omega(t+t'))\ dz \\
&\geq \widetilde{A} \mathbf{p}(st-1, \omega(0), \omega(t)) \mathbf{p}(st'-1, \omega(t), \omega(t+t')).
\end{aligned}$$

For $0 < t_1 < t_2 < +\infty$, by (2.8), there exists a constant $C > 0$, depending on t_1, t_2 and the curvature bounds, such that for any $\mathbf{v} \in S\widetilde{M}$, $x, y \in \widetilde{M}$, any $t, t_1 \le t \le t_2$,

$$C \exp\left[-(\frac{1}{4t_1} + \frac{\sigma}{3\sqrt{2t_1}})d^2(x,y)\right] \le p_{\mathbf{v}}(t,x,y).$$

As a consequence, we have

$$\mathbb{E}\left(\sup_{1+\frac{1}{s}\le t\le 2+\frac{1}{s}} F(s,\omega,t)\right) \le (\frac{1}{4s} + \frac{\sigma}{3\sqrt{2s}})\mathbb{E}\left(\sup_{1+\frac{1}{s}\le t\le 2+\frac{1}{s}} d^2(\omega(0),\omega(t))\right) - \ln(C\widetilde{A}),$$

where the second expectation term is bounded by a multiple of its value in a hyperbolic space with curvature the lower bound curvature of M and is finite (cf. [**DGM**]). So by the Subadditive Ergodic Theorem applied to the subadditive cocycle $F(s,\omega,t)$, there exists $h_{\mathcal{L}}(s)$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega_+$, and for $\widetilde{\mathbf{m}}$-a.e. $\mathbf{v}$,

(2.9)

$$h_{\mathcal{L}}(s) = \lim_{t\to+\infty} -\frac{1}{t}\ln \mathbf{p}(st-1,\omega(0),\omega(t)) = \lim_{t\to+\infty} -\frac{1}{t}\int_{\widetilde{M}} \mathbf{p}_{\mathbf{v}}(t,x,y)\ln \mathbf{p}_{\mathbf{v}}(st-1,x,y)\, d\mathrm{Vol}(y).$$

Using the semi-group property of $\mathbf{p}$ and (2.6) again, we obtain that for $0 < a < 1$, $s_1, s_2 > 0$,

$$\begin{aligned}&\mathbf{p}((as_1 + (1-a)s_2)t - 1, \omega(0), \omega(t))\\ &\qquad \ge \widetilde{A}\mathbf{p}(as_1 t - 1, \omega(0), \omega(at))\mathbf{p}((1-a)s_2 t - 1, \omega(at), \omega(t)).\end{aligned}$$

It follows that $h_{\mathcal{L}}(\cdot)$ is a convex function on $\mathbb{R}_+$ and hence is continuous. This allows us to pick up a full measure set of ω such that (2.4) holds true for all positive s. Let D be a countable dense subset of $\mathbb{R}_+$. There is a measurable set $E \subset \Omega_+$ with $\mathbb{P}(E) = 1$ such that for $\omega \in E$, (2.9) holds true for any $s \in D$. Let $\omega \in \Omega_+$ be such an orbit. Given any $s_1 < s_2$, let $t > 0$ be large, then we have by (2.6) that

$$\mathbf{p}(s_1 t, \omega(0), \omega(t)) \le A^{(s_1 - s_2)t+1}\mathbf{p}(s_2 t - 1, \omega(0), \omega(t)).$$

So for $s' < s < s''(s', s'' \in D)$, and $\omega \in E$,

$$\begin{aligned} h_{\mathcal{L}}(s'') + (s''-s)\ln A &\le \liminf_{t\to+\infty} -\frac{1}{t}\ln \mathbf{p}(st,\omega(0),\omega(t))\\ &\le \limsup_{t\to+\infty} -\frac{1}{t}\ln \mathbf{p}(st,\omega(0),\omega(t))\\ &\le h_{\mathcal{L}}(s') - (s-s')\ln A.\end{aligned}$$

Letting s', s'' go to s on both sides, it gives (2.4) by continuity of the function $h_{\mathcal{L}}$. Moreover, given $\omega \in E$, the convergence is uniform for s in any closed interval $[s_1, s_2], 0 < s_1 < s_2 < +\infty$.

To show the first equality in (2.5), we use the observation that for any $t \in \mathbb{R}_+$,

$$\mathbf{G}(\omega(0), \omega(t)) = t\int_0^{+\infty} \mathbf{p}(st, \omega(0), \omega(t))\; ds.$$

Let $s_0 \in (0,\infty)$ such that $h_{\mathcal{L}}(s_0) = \inf_{s>0} h_{\mathcal{L}}(s)$. For any $\varepsilon > 0$, there exists $\delta, 0 < \delta \le \varepsilon$, such that for $|s-s_0| < \delta, h_{\mathcal{L}}(s) \le h_{\mathcal{L}}(s_0) + \varepsilon$. Write

$$\mathbf{G}(\omega(0),\omega(t)) \ge t \int_{s_0+\frac{1}{t}}^{s_0+\delta} \mathbf{p}(st,\omega(0),\omega(t))\ ds$$

and note that for $s_0 + \frac{1}{t} < s < s_0+\delta$, $\omega \in \Omega_+$, we have as above by (2.6) that

$$\mathbf{p}(st,\omega(0),\omega(t)) \ge A^{(s-s_0)t+1}\mathbf{p}(s_0t-1,\omega(0),\omega(t)).$$

Moreover, for t large enough and $\omega \in E$, $\mathbf{p}(s_0t-1,\omega(0),\omega(t)) \ge e^{-t(h_{\mathcal{L}}(s_0)+\varepsilon)}$. Therefore:

$$(\mathbf{G}(\omega(\mathbf{0}),\omega(\mathbf{t})))^{1/t} \ge t^{1/t}A^{1/t}\left(\int_{1/t}^{\delta} A^{st}ds\right)^{1/t} e^{-(h_{\mathcal{L}}(s_0)+\varepsilon)}.$$

It follows that for $\omega \in E$,

$$\limsup_{t\to+\infty} -\frac{1}{t}\ln \mathbf{G}(\omega(0),\omega(t)) \le \inf_{s>0}\{h_{\mathcal{L}}(s)\}.$$

For the reverse inequality, we cut the integral $\int_0^{+\infty}\mathbf{p}(st,\omega(0),\omega(t))\ ds$ into three parts. Fix $\varepsilon_1 \in (0,h_{\mathcal{L}})$. We first claim that for $s_1 > 0$ small enough, for $\mathbb{P}$-a.e. paths $\omega \in \Omega_+$ and t large enough,

$$\int_0^{s_1}\mathbf{p}(st,\omega(0),\omega(t))\ ds \le \frac{1}{t}e^{-(\inf_{s>0}\{h_{\mathcal{L}}(s)\}-\varepsilon_1)t}. \tag{2.10}$$

Indeed, by Lemma 2.7, there exists a constant C' such that

$$\begin{aligned}
\int_0^{s_1}\mathbf{p}(st,\omega(0),\omega(t))\ ds &\le C'e^{C't}\int_0^{s_1}\frac{1}{(st)^{m/2}}e^{-\frac{d^2(\omega(0),\omega(t))}{C'st}}\ ds\\
&= \frac{C'e^{C't}}{t}\int_{1/(s_1t)}^{+\infty}u^{m/2+2}e^{-\frac{d^2(\omega(0),\omega(t))}{C'}u}\ du\\
&\le \frac{C'e^{C't}}{t}Q\left(d^2(\omega(0),\omega(t))\right)\ e^{-\frac{d^2(\omega(0),\omega(t))}{C's_1t}}, \qquad (2.11)
\end{aligned}$$

where Q is some polynomial of degree $[m/2]+3$. For $\mathbb{P}$-a.e. paths $\omega \in \Omega_+$, for large enough t,

$$0 < \frac{\ell_{\mathcal{L}}}{2} \le \frac{1}{t}d(\omega(0),\omega(t)) \le \frac{3\ell_{\mathcal{L}}}{2}.$$

It follows that for those paths, given $\varepsilon_1 \in (0,h_{\mathcal{L}})$, for any $s_1 \in (0, \frac{\ell_{\mathcal{L}}^2}{4C'}\cdot\frac{1}{C'+h_{\mathcal{L}}-\frac{1}{2}\varepsilon_1})$, the quantity in (2.11) is bounded from above by

$$\frac{1}{t}\cdot C'Q\left(d^2(\omega(0),\omega(t))\right)\cdot e^{-(\inf_{s>0}\{h_{\mathcal{L}}(s)\}-\frac{1}{2}\varepsilon_1)t}.$$

Consequently, (2.10) is satisfied for those paths, for t large enough.

Then observe that for $s_2, t > 1$, we have by (2.7) that

$$\int_{s_2}^{+\infty} \mathbf{p}(st, \omega(0), \omega(t))\ ds \leq B \int_{s_2}^{+\infty} e^{-\varepsilon st}\ ds = \frac{1}{\varepsilon t} B e^{-\varepsilon s_2 t}.$$

So for any $\varepsilon_1 \in (0, h_{\mathcal{L}})$, if s_2 and t are large enough, then

$$\int_{s_2}^{+\infty} \mathbf{p}(st, \omega(0), \omega(t))\ ds \leq \frac{1}{t} e^{-(\inf_{s>0}\{h_{\mathcal{L}}(s)\} - \varepsilon_1)t}.$$

Moreover, using the uniform convergence in (2.4) on the interval $[s_1, s_2]$, we get, for $\omega \in E$ and t large enough,

$$\begin{aligned} \int_{s_1}^{s_2} \mathbf{p}(st, \omega(0), \omega(t))\, ds &\leq (s_2 - s_1) e^{-\left(\inf_{s>0}\{h_{\mathcal{L}}(s)\} - \frac{1}{2}\varepsilon_1\right)t} \\ &\leq e^{-(\inf_{s>0}\{h_{\mathcal{L}}(s)\} - \varepsilon_1)t}. \end{aligned}$$

Putting everything together, we obtain

$$\liminf_{t \to +\infty} -\frac{1}{t} \ln \mathbf{G}(\omega(0), \omega(t)) \geq \inf_{s>0}\{h_{\mathcal{L}}(s)\}.$$

Finally, we have $\inf_{s>0}\{h_{\mathcal{L}}(s)\} \geq h_{\mathcal{L}}$ since for any typical $\mathbf{v} \in SM$,

$$\begin{aligned} h_{\mathcal{L}}(s) - h_{\mathcal{L}} &= \lim_{t \to +\infty} -\frac{1}{t} \int p_{\mathbf{v}}(t, x, y) \ln \frac{p_{\mathbf{v}}(st, x, y)}{p_{\mathbf{v}}(t, x, y)}\ d\mathrm{Vol}(y) \\ &\geq \lim_{t \to +\infty} \frac{1}{t} \int p_{\mathbf{v}}(t, x, y) \left(1 - \frac{p_{\mathbf{v}}(st, x, y)}{p_{\mathbf{v}}(t, x, y)}\right)\ d\mathrm{Vol}(y) \\ &\geq 0. \end{aligned}$$

□

2.4. **Linear drift and stochastic entropy for laminated diffusions: integral formulas.** The interrelation between the underlying geometry of the manifold and the linear drift and the stochastic entropy is not well exposed in the pathwise limit expressions (2.1) and (2.3). The purpose of this subsection is to establish the generalization of formulas (1.2) for the linear drift and the stochastic entropy, respectively, and set up the corresponding notations.

We begin with $\ell_{\mathcal{L}}$. We will express it using the Busemann function at the geometric boundary and the $\mathcal{L}$-harmonic measure. Recall the geometric boundary $\partial\widetilde{M}$ of $\widetilde{M}$ is the collection of equivalent classes of geodesics, where two geodesics γ_1, γ_2 of $\widetilde{M}$ are said to be equivalent (or asymptotic) if $\sup_{t\geq 0} d(\gamma_1, \gamma_2) < +\infty$. Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and $\mathrm{pr}(-\langle \overline{X}, Y\rangle) > 0$. For $\mathbb{P}$-a.e. paths $\omega \in \Omega_+$, $\omega(t)$ converges to the geometric boundary as t goes to infinity ([**H2**]), where we still denote by ω its projection to $\widetilde{M}$. Let $\gamma_{\omega(0),\omega(\infty)}$ be the geodesic ray starting from $\omega(0)$ asymptotic to $\omega(\infty) := \lim_{t\to+\infty} \omega(t)$. Then, loosely speaking, ω stays close to $\gamma_{\omega(0),\omega(\infty)}$ (see Lemma 3.5). The

Busemann function to be introduced will be very helpful to record the movement of the 'shadow' of $\omega(t)$ on $\gamma_{\omega(0),\omega(\infty)}$.

Let $x \in \widetilde{M}$ and define for $y \in \widetilde{M}$ the *Busemann function* $b_{x,y}(z)$ on $\widetilde{M}$ by letting

$$b_{x,y}(z) := d(y,z) - d(y,x), \text{ for } z \in \widetilde{M}.$$

The assignment of $y \mapsto b_{x,y}$ is continuous, one-to-one and takes value in a relatively compact set of functions for the topology of uniform convergence on compact subsets of $\widetilde{M}$. The Busemann compactification of $\widetilde{M}$ is the closure of $\widetilde{M}$ for that topology. In the negative curvature case, the Busemann compactification coincides with the geometric compactification (see [**Ba**]). So for each $\mathbf{v} = (x,\xi) \in \widetilde{M} \times \partial\widetilde{M}$, the *Busemann function at* $\mathbf{v}$, given by

$$b_{\mathbf{v}}(z) := \lim_{y\to\xi} b_{x,y}(z), \text{ for } z \in \widetilde{M},$$

is well-defined. For points on the geodesic $\gamma_{x,\xi}$, its Busemann function value is negative its flow distance with x. In other words, for $s, t \geq 0$,

$$b_{\mathbf{v}}(\gamma_{x,\xi}(t)) - b_{\mathbf{v}}(\gamma_{x,\xi}(s)) = s - t. \tag{2.12}$$

The equation (2.12) continues to hold if we replace $\gamma_{x,\xi}$ with geodesic $\gamma_{z,\xi}$ starting from $z \in \widetilde{M}$ which is asymptotic to ξ ([**EO**]). Note that the absolute value of the difference of the Busemann function at two points are always less than their distance. It follows that, if we consider the Busemann function $b_{\mathbf{v}}$ as a function defined on $W^s(x,\xi)$,

$$\nabla b_{\mathbf{v}}(z) = -\overline{X}(z,\xi), \tag{2.13}$$

where $\overline{X}(z,\xi)$ is the tangent vector to $W^s(\mathbf{v})$ which projects to $(z,\xi) = \dot{\gamma}_{z,\xi}(0)$. This relationship explains why the Busemann function is involved in the analysis of geometric and dynamical quantities: the variation of $\overline{X}$ is related to variation of asymptotic geodesics, the theory of Jacobi fields; while the vector field $\overline{X}$ on $S\widetilde{M}$ defines the geodesic flow.

We are going to use both interpretations of $\overline{X}$ to see how the linear drift is related the geometry. Since we only discuss C^3 metrics in this paper, we will state the results in this setting. But most results have corresponding versions for C^k metrics.

We begin with the theory of Jacobi fields and Jacobi tensors. Most notations will also be used in Section 4. Recall the Jacobi fields along a geodesic γ are vector fields $t \mapsto J(t) \in T_{\gamma(t)}M$ which describe infinitesimal variation of geodesics around γ. It is well-known that $J(t)$ satisfies the Jacobi equation

$$\nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)}J(t) + R(J(t),\dot{\gamma}(t))\dot{\gamma}(t) = 0 \tag{2.14}$$

and is uniquely determined by the values of $J(0)$ and $J'(0)$. (Here for vector fields Y, Z along $\widetilde{M}$, we denote $\nabla_Y Z$ and $R(Y,Z)$ the *covariant derivative* and the *curvature tensor* associated to the Levi-Civita connection of $\widetilde{g}$.) Let $N(\gamma)$ be the normal bundle of γ:

$$N(\gamma) := \cup_{t\in\mathbb{R}} N_t(\gamma), \text{ where } N_t(\gamma) = \{Y \in T_{\gamma(t)}M : \langle Y, \dot{\gamma}(t)\rangle = 0\}.$$

It follows from (2.14) that if $J(0)$ and $J'(0)$ both belong to $N_0(\gamma)$, then $J(t)$ and $J'(t)$ both belong to $N_t(\gamma)$, for all $t \in \mathbb{R}$. Also, it is easy to deduce from (2.14) that the Wronskian of two Jacobi fields J and $\widetilde{J}$ along γ:

$$W(J, \widetilde{J}) := \langle J', \widetilde{J} \rangle - \langle J, \widetilde{J}' \rangle$$

is constant.

A $(1,1)$-*tensor along* γ is a family $V = \{V(t),\ t \in \mathbb{R}\}$, where $V(t)$ is an endomorphism of $N_t(\gamma)$ such that for any family Y_t of parallel vectors along γ, the covariant derivative $\nabla_{\dot\gamma(t)}(V(t)Y_t)$ exists. The curvature tensor R induces a symmetric $(1,1)$-tensor along γ by $R(t)Y = R(Y, \dot\gamma(t))\dot\gamma(t)$. A $(1,1)$-tensor $V(t)$ along γ is called a *Jacobi tensor* if it satisfies

$$\nabla_{\dot\gamma(t)} \nabla_{\dot\gamma(t)} V(t) + R(t)V(t) = 0.$$

If $V(t)$ is a Jacobi tensor along γ, then $V(t)Y_t$ is a Jacobi field for any parallel field Y_t.

For each $s > 0$, $\mathbf{v} \in S\widetilde{M}$, let $S_{\mathbf{v},s}$ be the Jacobi tensor along the geodesic $\gamma_{\mathbf{v}}$ with the boundary conditions $S_{\mathbf{v},s}(0) = \mathrm{Id}$ and $S_{\mathbf{v},s}(s) = 0$. Since $(\widetilde{M}, \widetilde{g})$ has no conjugate points, the limit $\lim_{s \to +\infty} S_{\mathbf{v},s} =: S_{\mathbf{v}}$ exists ([**Gre**]). The tensor $S_{\mathbf{v}}$ is called the *stable tensor* along the geodesic $\gamma_{\mathbf{v}}$. Similarly, by reversing the time s, we obtain the *unstable tensor* $U_{\mathbf{v}}$ along the geodesic $\gamma_{\mathbf{v}}$.

To relate the stable and unstable tensors to the dynamics of the geodesic flow, we first recall the metric structure of the tangent space $TT\widetilde{M}$ of $T\widetilde{M}$. For $x \in \widetilde{M}$ and $\mathrm{v} \in T_x\widetilde{M}$, an element $w \in T_{\mathrm{v}}T\widetilde{M}$ is *vertical* if its projection on $T_x\widetilde{M}$ vanishes. The vertical subspace V_{v} is identified with $T_x\widetilde{M}$. The connection defines a *horizontal* complement H_{v}, also identified with $T_x\widetilde{M}$. This gives a horizontal/vertical Whitney sum decomposition

$$TT\widetilde{M} = T\widetilde{M} \oplus T\widetilde{M}.$$

Define the inner product on $TT\widetilde{M}$ by

$$\langle (Y_1, Z_1), (Y_2, Z_2) \rangle_{\widetilde{g}} := \langle Y_1, Y_2 \rangle_{\widetilde{g}} + \langle Z_1, Z_2 \rangle_{\widetilde{g}}.$$

It induces a Riemannian metric on $T\widetilde{M}$, the so-called Sasaki metric. The unit tangent bundle $S\widetilde{M}$ of the universal cover $(\widetilde{M}, \widetilde{g})$ is a subspace of $T\widetilde{M}$ with tangent space

$$T_{(x,\mathrm{v})}S\widetilde{M} = \{(Y, Z) :\ Y, Z \in T_x\widetilde{M}, Z \perp \mathrm{v}\},\ \text{ for } x \in \widetilde{M}, \mathrm{v} \in S_x\widetilde{M}.$$

Assume $\mathbf{v} = (x, \mathrm{v}) \in S\widetilde{M}$. Horizontal vectors in $T_{\mathbf{v}}S\widetilde{M}$ correspond to pairs $(J(0), 0)$. In particular, the geodesic spray $\overline{X}_{\mathbf{v}}$ at $\mathbf{v}$ is the horizontal vector associated with $(\mathrm{v}, 0)$. A vertical vector in $T_{\mathbf{v}}S\widetilde{M}$ is a vector tangent to $S_x\widetilde{M}$. It corresponds to a pair $(0, J'(0))$, with $J'(0)$ orthogonal to v. The orthogonal space to $\overline{X}_{\mathbf{v}}$ in $T_{\mathbf{v}}SM$ corresponds to pairs $(\mathrm{v}_1, \mathrm{v}_2), \mathrm{v}_i \in N_0(\gamma_{\mathbf{v}})$ for $i = 1, 2$.

The dynamical feature of Jacobi fields can be seen using the geodesic flow on the unit tangent bundle. Let $\mathbf{\Phi}_t$ be the time t map of the geodesic flow on $S\widetilde{M}$, in coordinates,

$$\mathbf{\Phi}_t(x,\xi) = (\gamma_{x,\xi}(t), \xi),\ \forall (x,\xi) \in S\widetilde{M}.$$

Let $D\mathbf{\Phi}_t$ be the tangent map of $\mathbf{\Phi}_t$. Then, if $(J(0), J'(0))$ is the horizontal/vertical decomposition of $\mathbf{w} \in T_{(x,\xi)}S\widetilde{M}$, $(J(t), J'(t))$ is the horizontal/vertical decomposition of $D\mathbf{\Phi}_t\mathbf{w} \in T_{\mathbf{\Phi}_t(x,\xi)}S\widetilde{M}$.

Due to the negative curvature nature of the metric, the geodesic flow on $S\widetilde{M}$ is *Anosov*: the tangent bundle $TS\widetilde{M}$ decomposes into the Witney sum of three $D\mathbf{\Phi}_t$-invariant subbundles $\mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{ss}} \oplus \mathbf{E}^{\mathrm{uu}}$, where $\mathbf{E}^{\mathrm{c}}$ is the 1-dimensional subbundle tangent to the flow and $\mathbf{E}^{\mathrm{ss}}$ and $\mathbf{E}^{\mathrm{uu}}$ are the strongly contracting and expanding subbundles, respectively, so that there are constants $C, c > 0$ such that

i) $\|D\mathbf{\Phi}_t\mathbf{w}\| \le Ce^{-ct}\|\mathbf{w}\|$ for $\mathbf{w} \in \mathbf{E}^{\mathrm{ss}}$, $t > 0$.

ii) $\|D\mathbf{\Phi}_t^{-1}\mathbf{w}\| \le Ce^{-ct}\|\mathbf{w}\|$ for $\mathbf{w} \in \mathbf{E}^{\mathrm{uu}}$, $t > 0$.

For any geodesic $\mathbf{v} = (x,\xi) \in S\widetilde{M}$, let $S_{\mathbf{v}}, U_{\mathbf{v}}$ be the stable and unstable tensors along $\gamma_{\mathbf{v}}$, respectively. The stable subbundle $\mathbf{E}^{\mathrm{ss}}$ at $\mathbf{v}$ is the graph of the mapping $S'_{\mathbf{v}}(0)$, considered as a map from $\overline{N_0(\gamma_{\mathbf{v}})}$ to $V_{\mathbf{v}}$ sending Y to $S'_{\mathbf{v}}(0)Y$, where $\overline{N_0(\gamma_{\mathbf{v}})} := \{\mathbf{w}, \mathbf{w} \in H_{\mathbf{v}}, \mathbf{w} \perp \overline{X}_{\mathbf{v}}\}$. Similarly, the unstable subbundle $\mathbf{E}^{\mathrm{uu}}$ at $\mathbf{v}$ is the graph of the mapping $U'_{\mathbf{v}}(0)$ considered as a map from $\overline{N_0(\gamma_{\mathbf{v}})}$ to $V_{\mathbf{v}}$. Due to the Anosov property of the geodesic flow, the distributions of $\mathbf{E}^{\mathrm{ss}}, \mathbf{E}^{\mathrm{uu}}$ (and hence $\mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{ss}}, \mathbf{E}^{\mathrm{c}} \oplus \mathbf{E}^{\mathrm{uu}}$) are Hölder continuous (this is first proved by Anosov ([**Ano2**]), see [**Ba**, Proposition 4.4] for a similar but simpler argument by Brin). As a consequence, the $(1,1)$-tensors $S_{\mathbf{v}}, S'_{\mathbf{v}}, U_{\mathbf{v}}, U'_{\mathbf{v}}$ are also Hölder continuous with respect to $\mathbf{v}$.

We are in a situation to see the relation between the Busemann function and the geodesic flow. Let $x_0 \in \widetilde{M}$ be a reference point and for any $\xi \in \partial\widetilde{M}$ consider $b_{x_0,\xi}(\cdot) := \lim_{z\to\xi} b_{x_0,z}(\cdot)$. For any $\mathbf{v} = (x,\xi) \in \widetilde{M} \times \partial\widetilde{M}$, the set

$$\{(y,\xi) :\ b_{x_0,\xi}(y) = b_{x_0,\xi}(x)\}$$

turns out to coincide with the *strong stable manifold at* $\mathbf{v}$, denoted $W^{ss}(\mathbf{v})$, which is

$$W^{ss}(\mathbf{v}) := \left\{(y,\eta) :\ \limsup_{t\to+\infty} \frac{1}{t} \log \operatorname{dist}\left(\Phi_t(y,\eta), \Phi_t(\mathbf{v})\right) < 0\right\}.$$

(The *strong unstable manifold at* $\mathbf{v}$, denoted $W^{su}(\mathbf{v})$, is defined by reversing the time.) In other words, the collection of the foot points y such that $(y,\xi) \in W^{ss}(x,\xi)$ form the *stable horosphere*, which is a level set of Busemann function. Note that $W^{ss}(\mathbf{v})$ locally is a C^2 graph from $\mathbf{E}^{\mathrm{ss}}_{\mathbf{v}}$ to $\mathbf{E}^{\mathrm{c}}_{\mathbf{v}} \oplus \mathbf{E}^{\mathrm{uu}}_{\mathbf{v}}$ and is tangent to $\mathbf{E}^{\mathrm{ss}}_{\mathbf{v}}$. So, by the Jacobian characterization of $\mathbf{E}^{\mathrm{ss}}_{\mathbf{v}}$ of the previous paragraph and (2.13), it is true ([**Esc**, **HIH**]) that

$$\nabla_{\mathbf{w}}(\nabla b_{x,\xi})(x) = -S'_{(x,\xi)}(0)(\mathbf{w}),\ \forall \mathbf{w} \in T_x\widetilde{M}.$$

Thus,

$$\Delta_x b_{x,\xi} = -\mathrm{Div}\overline{X} = -\text{Trace of } S'_{(x,\xi)}(0),$$

which is the mean curvature of the horosphere $W^{ss}(x,\xi)$ at x. Note that for each $\psi \in G$,

$$b_{x_0,\psi\xi}(\psi x) = b_{x_0,\xi}(x) + b_{\psi^{-1}x_0,\xi}(x_0).$$

Hence $\Delta_x b_{x_0,\xi}$ satisfies $\Delta_{\psi x} b_{x_0,\psi\xi} = \Delta_x b_{x_0,\xi}$ and defines a function B on the unit tangent bundle SM, which is called the *Laplacian of the Busemann function*. Due to the hyperbolic nature of the geodesic flow, the function B is a Hölder continuous function on SM.

Now, we can state the integral formula for the linear drift.

Proposition 2.9. *Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and $pr(-\langle \overline{X}, Y\rangle) > 0$. Then we have*

$$\ell_{\mathcal{L}} \;=\; -\int_{M_0\times\partial\widetilde{M}} \left(\mathrm{Div}\overline{X} + \langle Y, \overline{X}\rangle\right)\, d\widetilde{\mathbf{m}}. \tag{2.15}$$

(Observe that the classical formula (1.1) for the linear drift is obtained from Proposition 2.9 by considering the metric g^λ and $Y \equiv 0$.)

Proof. For $\mathbb{P}$-a.e. path $\omega \in \Omega_+$, we still denote ω its projection to $\widetilde{M}$ and let $\mathbf{v} := \omega(0)$ and $\eta := \lim_{t\to+\infty}\omega(t) \in \partial\widetilde{M}$. We see that when t goes to infinity, the process $b_{\mathbf{v}}(\omega(t)) - d(x,\omega(t))$ converges $\mathbb{P}$-a.e. to the a.e. finite number $-2(\xi|\eta)_x$, where

$$(\xi|\eta)_x := \lim_{y\to\xi, z\to\eta}(y|z)_x \text{ and } (y|z)_x := \frac{1}{2}\left(d(x,y) + d(x,z) - d(y,z)\right). \tag{2.16}$$

So for $\mathbb{P}$-a.e. $\omega \in \Omega_+$, we have

$$\lim_{t\to+\infty}\frac{1}{t} b_{\mathbf{v}}(\omega(t)) = \ell_{\mathcal{L}}.$$

Using the fact that the $\mathcal{L}$-diffusion has leafwise infinitesimal generator $\Delta + Y$ and is ergodic with invariant measure $\mathbf{m}$ on SM, we obtain

$$\begin{aligned}
\ell_{\mathcal{L}} &= \lim_{t\to+\infty}\frac{1}{t}\int_0^t \frac{\partial}{\partial s} b_{\mathbf{v}}(\omega(s))\, ds \\
&= \lim_{t\to+\infty}\frac{1}{t}\int_0^t (\Delta + Y) b_{\mathbf{v}}(\omega(s))\, ds \left(= \int_{M_0\times\partial\widetilde{M}} (\Delta + Y) b_{\mathbf{v}}\ d\widetilde{\mathbf{m}}\right) \\
&= -\int_{M_0\times\partial\widetilde{M}} \left(\mathrm{Div}\overline{X} + \langle Y, \overline{X}\rangle\right)\, d\widetilde{\mathbf{m}}.
\end{aligned}$$

□

The negative of the logarithm of the Green function has a lot of properties analogous to a distance function. First of all, let us recall some classical results concerning Green functions from [**Anc**].

Lemma 2.10. *(see* [**Anc**, Remark 3.1]*) Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and $pr(-\langle \overline{X}, Y\rangle) > 0$ and let $\mathbf{G}(\cdot,\cdot) = \{G_{\mathbf{v}}(\cdot,\cdot)\}_{\mathbf{v}\in S\widetilde{M}}$ be the Green function of $\mathcal{L}$. There exists a constant $c_0 \in (0,1)$ such that for any $\mathbf{v} \in S\widetilde{M}$ and any $x, y, z \in \widetilde{M}$ with mutual distances greater than 1,*

$$G_{\mathbf{v}}(x,z) \geq c_0 G_{\mathbf{v}}(x,y) G_{\mathbf{v}}(y,z). \tag{2.17}$$

For $\mathrm{v}, \mathrm{w} \in S_x\widetilde{M}$, $x \in \widetilde{M}$, the angle $\angle_x(\mathrm{v},\mathrm{w})$ is the unique number $0 \leq \theta \leq \pi$ such that $\langle v, w\rangle = \cos\theta$. Given $\mathrm{v} \in S_x\widetilde{M}$ and $0 < \theta < \pi$, the set

$$\Gamma_x(\mathrm{v},\theta) := \{y \in \widetilde{M} \cup \partial\widetilde{M} : \quad \angle_x(\mathrm{v}, \dot{\gamma}_{x,y}(0)) < \theta\}$$

is called the *cone of vertex x, axis* v*, and angle θ*, where $\gamma_{x,y}$ is the geodesic segment that starts at x and ends at y. For any $s > 0$, the cone Γ with vertex $\gamma_{\mathrm{v}}(s)$ (where γ_{v} is the geodesic starting at x with initial speed v), axis $\dot{\gamma}_{\mathrm{v}}(s)$ and angle θ is called the *s-shifted cone* of $\Gamma_x(\mathrm{v},\theta)$. The following is a special case of the Ancona's inequality at infinity ([**Anc**]).

Lemma 2.11. *(see* [**Anc**, Theorem 1']*) Let $\mathcal{L}$ and $\mathbf{G}$ be as in Lemma 2.10. Let $\Gamma := \Gamma_{x_0}(\mathrm{v}, \frac{\pi}{2})$ be a cone in $\widetilde{M}$ with vertex x_0, axis* v *and angle $\frac{\pi}{2}$. Let Γ_1 be the* 1*-shifted cone of Γ and x_1 be the vertex of Γ_1. There exists a constant c_1 such that for any $\mathbf{v} \in S\widetilde{M}$, any Γ, all $x \in \widetilde{M}\backslash\Gamma$ and $z \in \Gamma_1$,*

$$G_{\mathbf{v}}(x,z) \leq c_1 G_{\mathbf{v}}(x,x_1) G_{\mathbf{v}}(x_0,z). \tag{2.18}$$

We may assume $c_1 = c_0^{-1}$, where c_0 is as in Lemma 2.10. As a consequence of Lemma 2.10 and Lemma 2.11, $\mathbf{G}$ is related to the distance d in the following way.

Lemma 2.12. *Let $\mathcal{L}$ and $\mathbf{G}$ be as in Lemma 2.10. There exist positive numbers $c_2, c_3, \alpha_2, \alpha_3$ such that for any $\mathbf{v} \in S\widetilde{M}$ and any $x, z \in \widetilde{M}$ with $d(x,z) \geq 1$,*

$$c_2 e^{-\alpha_2 d(x,z)} \leq G_{\mathbf{v}}(x,z) \leq c_3 e^{-\alpha_3 d(x,z)}. \tag{2.19}$$

Proof. The upper bound of (2.19) was shown in [**H2**, Corollary 4.8] using Ancona's inequality at infinity (cf. Lemma 2.11). For the lower bound, we first observe that Lemma 2.10 also holds true if x, y, z satisfies $d(x,z) > 1$ and $d(x,y) = 1$. Indeed, by the classical Harnack inequality ([**LY**]), there exists $c_4 \in (0,1)$ such that for any $\mathbf{v} \in S\widetilde{M}$ and $x, y, z \in \widetilde{M}$ with $d(x,z) > 1$ and $d(x,y) \leq 1$,

$$c_4 G_{\mathbf{v}}(y,z) \leq G_{\mathbf{v}}(x,z) \leq c_4^{-1} G_{\mathbf{v}}(y,z). \tag{2.20}$$

Since $d(x,y) = 1$, by [**Anc**, Proposition 7], there is $c_5 \in (0,1)$ (independent of x, y) with

$$c_5 \leq G_{\mathbf{v}}(x,y) \leq c_5^{-1}. \tag{2.21}$$

So, if $c_0 \leq c_4 c_5$, then (2.17) holds true for $x, y, z \in \widetilde{M}$ with $d(x,z) > 1$ and $d(x,y) = 1$. Now, for $x, z \in \widetilde{M}$ with $d(x,z) > 1$, choose a sequence of points $x_i, 1 \leq i \leq n$, on the geodesic segment $\gamma_{x,z}$ with $x_0 = x, x_n = z$, $d(x_i, x_{i+1}) = 1, i = 0, \cdots, n-2$, and $d(x_{n-1}, z) \in [1, 2)$. Applying (2.17) successively for x_i, x_{i+1}, z, we obtain

$$G_{\mathbf{v}}(x,z) \geq G_{\mathbf{v}}(x_{n-1}, z)(c_0c_5)^{n-1} \geq c_4c_5(c_0c_5)^{n-1} \geq c_4c_5(c_0c_5)^{d(x,y)},$$

where, to derive the second inequality, we use (2.20) and the fact that the lower bound of (2.21) holds for any $x, y \in \widetilde{M}$ with $d(x,y) \leq 1$. The lower bound estimation of (2.19) follows for $c_2 = c_4c_5$ and $\alpha_2 = -\ln c_0c_5$. □

We may assume the constants c_2, c_3 in Lemma 2.12 are such that c_2 is smaller than 1 and $c_3 = c_2^{-1}$. For each $\mathbf{v} \in S\widetilde{M}$, $x, z \in \widetilde{M}$, let

$$d_{G_{\mathbf{v}}}(x,z) := \begin{cases} -\ln\left(c_2 G_{\mathbf{v}}(x,z)\right), & \text{if } d(x,z) > 1; \\ -\ln c_2, & \text{otherwise.} \end{cases}$$

Although $d_{G_{\mathbf{v}}}$ is always greater than the positive number $\min\{\alpha_3, -\ln c_2\}$ by (2.19), we still call it a *'Green metric'* for $\mathcal{L}_{\mathbf{v}}$ (after [**BHM**] for the hyperbolic groups case) since it satisfies an almost triangle inequality in the following sense.

Lemma 2.13. *There exists a constant $c_6 \in (0,1)$ such that for all $x, y, z \in \widetilde{M}$,*

$$d_{G_{\mathbf{v}}}(x,z) \leq d_{G_{\mathbf{v}}}(x,y) + d_{G_{\mathbf{v}}}(y,z) - \ln c_6. \tag{2.22}$$

Proof. If $d(x,z) \leq 1$, then (2.22) holds for $c_6 = c_2$. If x, y, z have mutual distances greater than 1, then (2.22) holds for $c_6 = c_0$ by Lemma 2.10. If $d(x,z) > 1$ and $d(y,z) \leq 1$, using the classical Harnack inequality (2.20), we have

$$G_{\mathbf{v}}(x,z) \geq c_4 G_{\mathbf{v}}(x,y)$$

and hence (2.22) holds with $c_6 = c_4$ if, furthermore, $d(x,y) > 1$ or with $c_6 = c_4c_5$ otherwise. The case that $d(x,z) > 1$, $d(x,y) \leq 1$ can be treated similarly. □

By Lemma 2.12, $d_{G_{\mathbf{v}}}$ is comparable to the metric d for any $x, z \in \widetilde{M}$ with $d(x,z) > 1$:

$$\alpha_3 d(x,z) \leq d_{G_{\mathbf{v}}}(x,z) \leq \alpha_2 d(x,z) - 2\ln c_2. \tag{2.23}$$

Using Lemma 2.11, we can further obtain that $d_{G_{\mathbf{v}}}$ is almost additive along the geodesics.

Lemma 2.14. *Let $\mathcal{L}$ and $\mathbf{G}$ be as in Lemma 2.10. There exists a constant c_7 such that for any $\mathbf{v} \in S\widetilde{M}$, any $x, z \in \widetilde{M}$ and y in the geodesic segment $\gamma_{x,z}$ connecting x and z,*

$$|d_{G_{\mathbf{v}}}(x,y) + d_{G_{\mathbf{v}}}(y,z) - d_{G_{\mathbf{v}}}(x,z)| \leq -\ln c_7. \tag{2.24}$$

Proof. Let $x, z \in \widetilde{M}$ and y belong to the geodesic segment $\gamma_{x,z}$. If $d(x,y), d(y,z) \leq 1$, then $d(x,z) \leq 2$ and, using (2.23), we obtain (2.24) with $c_7 = c_2^2 e^{-2\alpha_2}$. If $d(x,y) \leq 1$ and $d(y,z) > 1$ (or $d(y,z) \leq 1$ and $d(x,y) > 1$), using Harnack's inequality (2.20), we have (2.24) with $c_7 = c_2c_4$. Finally, if x, y, z have mutual distances greater than 1, we have by Lemma 2.10 and Lemma 2.11 (where we can use Harnack's inequality to replace $G_{\mathbf{v}}(x, x_1)$ in (2.18) by $c_4^{-1}G_{\mathbf{v}}(x, x_0)$) that

$$|\ln G_{\mathbf{v}}(x,y) + \ln G_{\mathbf{v}}(y,z) - \ln G_{\mathbf{v}}(x,z)| \leq -\ln(c_1c_4)$$

and consequently,

$$|d_{G_{\mathbf{v}}}(x,y)+d_{G_{\mathbf{v}}}(y,z)-d_{G_{\mathbf{v}}}(x,z)|\le -\ln(c_1c_2c_4).$$

□

More is true, as we can see from Lemma 2.11 and Lemma 2.13 as well.

Lemma 2.15. *Let $\mathcal{L}$ and $\mathbf{G}$ be as in Lemma 2.10. There exists a constant c_8 such that for any $\mathbf{v}\in S\widetilde{M}$, if $x,y,z\in\widetilde{M}$ are such that x and z are separated by some cone Γ with vertex y and angle $\frac{\pi}{2}$, and Γ_1, the 1-shifted cone of Γ, i.e., $x\in\widetilde{M}\backslash\Gamma$, $z\in\Gamma_1$, then*

$$|d_{G_{\mathbf{v}}}(x,y)+d_{G_{\mathbf{v}}}(y,z)-d_{G_{\mathbf{v}}}(x,z)|\le -\ln c_8.$$

The counterpart of the Busemann function for the analysis of the pathwise limits for stochastic entropy is the Poisson kernel function. Let $\mathbf{v}=(x,\xi)\in\widetilde{M}\times\partial\widetilde{M}$. A *Poisson kernel function* $k_{\mathbf{v}}(\cdot,\eta)$ of $\mathcal{L}_{\mathbf{v}}$ at $\eta\in\partial\widetilde{M}$ is a positive $\mathcal{L}_{\mathbf{v}}$-harmonic function on $\widetilde{M}$ such that

$$k_{\mathbf{v}}(x,\eta)=1, k_{\mathbf{v}}(y,\eta)=O(G_{\mathbf{v}}(x,y)),\ \text{as } y\to\eta'\neq\eta.$$

A point $\eta\in\partial\widetilde{M}$ is a *Martin point of* $\mathcal{L}_{\mathbf{v}}$ if it satisfies the following properties:

i) there exists a Poisson kernel function $k_{\mathbf{v}}(\cdot,\eta)$ of $\mathcal{L}_{\mathbf{v}}$ at η,
ii) the Poisson kernel function is unique, and
iii) if $y_n\to\eta$, then $\ln G_{\mathbf{v}}(\cdot,y_n)-\ln G_{\mathbf{v}}(x,y_n)\to\ln k_{\mathbf{v}}(\cdot,\eta)$ uniformly on compact sets.

Since (M,g) is negatively curved and $\mathcal{L}_{\mathbf{v}}$ is weakly coercive, every point η of the geometric boundary $\partial\widetilde{M}$ is a Martin point by Ancona ([**Anc**]). Hence $k_{\mathbf{v}}(\cdot,\eta)$ is also called the Martin kernel function at η.

The function $k_{\mathbf{v}}(\cdot,\eta)$ should be understood as a function on $W^s(\mathbf{v})$ for all η, i.e. it is identified with $k_{\mathbf{v}}(\varpi(\cdot),\eta)$, where $\varpi: S\widetilde{M}\to\widetilde{M}$ is the projection map. In case $\mathcal{L}=\Delta$, all the $k_{\mathbf{v}}(\cdot,\eta)$'s are the same as $k_\eta(\cdot)$, the Martin kernel function on $\widetilde{M}$ associated to Δ. In general, $k_{\mathbf{v}}$ may vary from leaf to leaf.

Finally, we can state the integral formula for the stochastic entropy.

Proposition 2.16. *Let $\mathcal{L}=\Delta+Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^*=0$ leafwisely and $pr(-\langle\overline{X},Y\rangle)>0$. Then we have*

$$h_{\mathcal{L}} \;=\; \int_{M_0\times\partial\widetilde{M}}\|\nabla\ln k_{\mathbf{v}}(x,\xi)\|^2\,d\widetilde{\mathbf{m}}. \tag{2.25}$$

(Since each $k_{\mathbf{v}}(\cdot,\eta)$ is a function on $W^s(\mathbf{v})$, in particular, when $\eta=\xi$, its gradient (for the lifted metric from $\widetilde{M}$ to $W^s(\mathbf{v})$) is a tangent vector to $W^s(\mathbf{v})$. We also observe that the classical formula (1.1) for the stochastic entropy is obtained from Proposition 2.16 by considering the metric g^λ and $Y\equiv 0$.)

Proof. For $\mathbb{P}$-a.e. path $\omega \in \Omega_+$, we still denote ω its projection to $\widetilde{M}$ and write $\mathbf{v} := \omega(0)$. When t goes to infinity, we see that

$$\limsup_{t\to+\infty} |\ln G_{\mathbf{v}}(x, \omega(t)) - \ln k_{\mathbf{v}}(\omega(t), \xi)| < +\infty,$$

Indeed, let z_t be the point on the geodesic ray $\gamma_{\omega(t),\xi}$ closest to x. Then, as $t \to +\infty$,

$$G_{\mathbf{v}}(x, \omega(t)) \asymp G_{\mathbf{v}}(z_t, \omega(t)) \asymp \frac{G_{\mathbf{v}}(y, \omega(t))}{G_{\mathbf{v}}(y, z_t)} \tag{2.26}$$

for all y on the geodesic going from $\omega(t)$ to ξ, where $\asymp$ means up to some multiplicative constant independent of t. The first $\asymp$ comes from Harnack inequality using the fact that $\sup_t d(x, z_t)$ is finite $\mathbb{P}$-almost everywhere. (For $\mathbb{P}$-a.e. $\omega \in \Omega_+$, $\eta = \lim_{t\to+\infty} \omega(t)$ differs from ξ and $d(x, z_t)$, as $t \to +\infty$, tends to the distance between x and the geodesic asymptotic to ξ and η in opposite directions.) The second $\asymp$ comes from Ancona's inequality ([**Anc**]). Replace $G_{\mathbf{v}}(y, \omega(t))/G_{\mathbf{v}}(y, z_t)$ by its limit as $y \to \xi$, which is $k_{(z_t,\xi)}(\omega(t), \xi)$, which is itself $\asymp k_{\mathbf{v}}(\omega(t), \xi)$ by Harnack inequality again. By (2.3), it follows that, for $\mathbb{P}$-a.e. $\omega \in \Omega_+$,

$$\lim_{t\to+\infty} -\frac{1}{t} \ln k_{\mathbf{v}}(\omega(t), \xi) = h_{\mathcal{L}}.$$

Again, using the fact that the $\mathcal{L}$-diffusion has leafwise infinitesimal generator $\Delta + Y$ and is ergodic, we obtain

$$\begin{aligned}
h_{\mathcal{L}} &= \lim_{t\to+\infty} -\frac{1}{t} \int_0^t \frac{\partial}{\partial s} (\ln k_{\mathbf{v}}(\omega(s), \xi))\ ds \\
&= \lim_{t\to+\infty} \frac{1}{t} \int_0^t -(\Delta + Y)\left(\ln k_{\mathbf{v}}(\omega(s), \xi)\right)\ ds \left(= -\int_{M_0\times\partial\widetilde{M}} (\Delta + Y)\left(\ln k_{\mathbf{v}}\right) d\widetilde{\mathbf{m}} \right) \\
&= \int_{M_0\times\partial\widetilde{M}} \|\nabla \ln k_{\mathbf{v}}(\cdot, \xi)\|^2\ d\widetilde{\mathbf{m}}.
\end{aligned}$$

The last equality comes from the fact that the Martin kernel function $k_{\mathbf{v}}(\cdot, \xi)$ satisfies $(\Delta + Y)(k_{\mathbf{v}}(\cdot, \xi)) = 0$. □

2.5. **A Central limit theorem for the linear drift and the stochastic entropy.** With the help of the Busemann function and the Martin kernel function, we can further describe the distributions of the pathwise limits for time large. In this subsection, we recall the Central Limit Theorems for $\ell_{\mathcal{L}}$ and $h_{\mathcal{L}}$ and the ingredients of the proof that we will use later.

Proposition 2.17. *([**H2**]) Let $\mathcal{L} = \Delta + Y$ be such that Y^*, the dual of Y in the cotangent bundle to the stable foliation over SM, satisfies $dY^* = 0$ leafwisely and $pr(-\langle \overline{X}, Y\rangle) > 0$. Then there are positive numbers σ_0 and σ_1 such that the distributions of the variables*

$$\frac{1}{\sigma_0\sqrt{t}} \left[d_{\mathcal{W}}(\omega(0), \omega(t)) - t\ell_{\mathcal{L}}\right] \quad \textit{and} \quad \frac{1}{\sigma_1\sqrt{t}} \left[\ln \mathbf{G}(\omega(0), \omega(t)) + th_{\mathcal{L}}\right]$$

are asymptotically close to the normal distribution when t goes to infinity.

The proof of the proposition relies on the contraction property of the action of the diffusion process on a certain space of Hölder continuous functions. Let Q_t $(t \geq 0)$ be the action of $[0, +\infty)$ on continuous functions f on SM which describes the $\mathcal{L}$-diffusion, i.e.,

$$Q_t(f)(x,\xi) = \int_{M_0 \times \partial \widetilde{M}} \widetilde{f}(y,\eta)\mathbf{p}(t,(x,\xi),d(y,\eta)),$$

where $\widetilde{f}$ denotes the G-invariant extension of f to $S\widetilde{M}$. For $\iota > 0$, define a norm $\|\cdot\|_\iota$ on the space of continuous functions f on SM by letting

$$\|f\|_\iota = \sup_{x,\xi} |\widetilde{f}(x,\xi)| + \sup_{x,\xi_1,\xi_2} |\widetilde{f}(x,\xi_1) - \widetilde{f}(x,\xi_2)| \exp(\iota(\xi_1|\xi_2)_x),$$

where $(\xi_1|\xi_2)_x$ is defined as in (2.16), and let $\mathcal{H}_\iota$ be the Banach space of continuous functions f on SM with $\|f\|_\iota < +\infty$. It was shown ([**H2**, Theorem 5.13]) that for sufficiently small $\iota > 0$, as $t \to \infty$, Q_t converges to the mapping $f \mapsto \int f \, d\mathbf{m}$ exponentially in t for $f \in \mathcal{H}_\iota$. As a consequence, one concludes that for any $f \in \mathcal{H}_\iota$ with $\int f \, d\mathbf{m} = 0$, $u = -\int_0^{+\infty} Q_t f \, dt$, is, up to an additive constant function, the unique element in $\mathcal{H}_\iota$ which solves $\mathcal{L}u = f$ ([**H2**, Corollary 5.14]). Applying this property to $b_{\mathbf{v}}$ and $k_{\mathbf{v}}(\cdot,\xi)$, where we observe that both $\mathbf{v} \mapsto \Delta b_{\mathbf{v}}$ and $\mathbf{v} \mapsto \nabla \ln k_{\mathbf{v}}(\cdot,\xi)$ are G-invariant and descend to Hölder continuous functions on SM (see [**Ano1**, **HPS**] and [**H1**], respectively), we obtain two Hölder continuous functions u_0, u_1 on SM (or on $M_0 \times \partial\widetilde{M}$) such that

$$\begin{aligned}
\mathcal{L}(u_0) &= -\left(\mathrm{Div}(\overline{X}) + \langle Y, \overline{X}\rangle\right) + \int_{M_0 \times \partial\widetilde{M}} \left(\mathrm{Div}(\overline{X}) + \langle Y, \overline{X}\rangle\right) \, d\widetilde{\mathbf{m}} \\
&= -\left(\mathrm{Div}(\overline{X}) + \langle Y, \overline{X}\rangle\right) - \ell_{\mathcal{L}}, \quad \text{by} \quad (2.15), \text{ and} \\
\mathcal{L}(u_1) &= \|\nabla \ln k_{\mathbf{v}}(\cdot,\xi)\|^2 - \int_{M_0 \times \partial\widetilde{M}} \|\nabla \ln k_{\mathbf{v}}(\cdot,\xi)\|^2 \, d\widetilde{\mathbf{m}} \\
&= \|\nabla \ln k_{\mathbf{v}}(\cdot,\xi)\|^2 - h_{\mathcal{L}}, \text{ by } \ (2.25),
\end{aligned}$$

where we continue to denote u_0 and u_1 their G-invariant extensions to $S\widetilde{M}$. For each $\omega \in \Omega_+$ belonging to a stable leaf of $S\widetilde{M}$, we also denote ω its projection to $\widetilde{M}$. Then for $f = -b_{\mathbf{v}} + u_0$ (or $\ln k_{\mathbf{v}}(\cdot,\xi) + u_1$), $f(\omega(t)) - f(\omega(0)) - \int_0^t (\mathcal{L}f)(\omega(s)) \, ds$ is a martingale with increasing process $2\|\nabla f\|^2(\omega(t)) \, dt$. In other words, we have the following.

Proposition 2.18. *(cf.* [**L2**, Corollary 3]*) For any $\mathbf{v} = (x,\xi)$, the process $(\mathbf{Z}_t^0)_{t \in \mathbb{R}_+}$ with $\omega(0) = \mathbf{v}$ [respectively, $(\mathbf{Z}_t^1)_{t \in \mathbb{R}_+}$ with $\omega(0) = \mathbf{v}$],*

$$\mathbf{Z}_t^0 := -b_{\omega(0)}(\omega(t)) + t\ell_{\mathcal{L}} + u_0(\omega(t)) - u_0(\omega(0))$$

$\big[$*respectively,*

$$\mathbf{Z}_t^1 := \ln k_{\mathbf{v}}(\omega(t),\xi) + th_{\mathcal{L}} + u_1(\omega(t)) - u_1(\omega(0))\big]$$

is a martingale with increasing process

$$2\|\overline{X} + \nabla u_0\|^2(\omega(t)) \, dt \quad [\text{respectively, } 2\|\nabla \ln k_{\mathbf{v}}(\cdot,\xi) + \nabla u_1\|^2(\omega(t)) \, dt].$$

The last ingredient in the proof of Proposition 2.17 is a Central Limit Theorem for martingales.

Lemma 2.19. *([**RY**, Chapter IV, Theorem 1.3]) Let $\mathsf{M} = (\mathsf{M}_t)_{t\geq 0}$ be a continuous, square-integrable centered martingale with respect to an increasing filtration $(\mathfrak{F}_t)_{t\geq 0}$ of a probability space, with stationary increments. Assume that $\mathsf{M}_0 = 0$ and*

$$\lim_{t\to+\infty} \mathbb{E}\left(\left|\frac{1}{t}\langle \mathsf{M}, \mathsf{M}\rangle_t - \sigma^2\right|\right) = 0 \tag{2.27}$$

for some real number σ^2, where $\langle \mathsf{M}, \mathsf{M}\rangle_t$ denotes the quadratic variation of M_t. Then the laws of $\mathsf{M}_t/\sqrt{t}$ converge in distribution to a centered normal law with variance σ^2.

Now we see that both $\mathbf{Z}_t^0$ and $\mathbf{Z}_t^1$ are continuous and square integrable. The respective average variances converge to, respectively, σ_0^2 and σ_1^2, where

$$\begin{aligned}
\sigma_0^2 &= 2\int_{M_0\times\partial\widetilde{M}} \|\overline{X} + \nabla u_0\|^2 \, d\widetilde{\mathbf{m}},\\
\sigma_1^2 &= 2\int_{M_0\times\partial\widetilde{M}} \|\nabla \ln k_{\mathbf{v}}(\cdot,\xi) + \nabla u_1\|^2 \, d\widetilde{\mathbf{m}}.
\end{aligned}$$

By Proposition 2.2, the $\mathcal{L}$-diffusion system is mixing, (2.27) holds for $\mathbf{Z}_t^0$ and $\mathbf{Z}_t^1$ with $\sigma = \sigma_0$ or σ_1, respectively. Hence both $(1/(\sigma_0\sqrt{t}))\mathbf{Z}_t^0$ and $(1/(\sigma_1\sqrt{t}))\mathbf{Z}_t^1$ will converge to the normal distribution as t tends to infinity. Note that in the proof of Propositions 2.9 and 2.16 we have shown that for $\mathbb{P}$-a.e. $\omega\in\Omega_+$, $b_{\mathbf{v}}(\omega(t)) - d_{\mathcal{W}}(\omega(0),\omega(t))$ converges to a finite number and that

$$\limsup_{t\to+\infty} |\ln G_{\mathbf{v}}(\omega(0),\omega(t)) - \ln k_{\mathbf{v}}(\omega(t),\xi)| < +\infty.$$

As a consequence, we see from Proposition 2.18 that $(1/(\sigma_0\sqrt{t}))\,[d_{\mathcal{W}}(\omega(0),\omega(t)) - t\ell_{\mathcal{L}}]$ and $(1/(\sigma_0\sqrt{t}))\mathbf{Z}_t^0$ (respectively, $(1/(\sigma_1\sqrt{t}))\,[\ln \mathbf{G}(\omega(0),\omega(t)) + th_{\mathcal{L}}]$ and $(1/(\sigma_1\sqrt{t}))\mathbf{Z}_t^1$) have the same asymptotical distribution, which is normal, when t goes to infinity.

2.6. **Construction of the diffusion processes.** So far, we know that both the linear drift and the stochastic entropy are quantities concerning the average behavior of diffusions and they can be evaluated along typical paths. To see how they vary when we change the generators of the diffusions from $\mathcal{L}$ to $\mathcal{L}+Z$ (which also fulfills the requirement of Proposition 2.9 (or Proposition 2.16)), our very first step is to understand the change of distributions of the corresponding diffusion processes on the path spaces. For this, we use techniques of stochastic differential equation (SDE) to construct on the same probability space all the diffusion processes.

We begin with the general theories of SDE on a smooth manifold $\mathbf{N}$. Let $X_1,\cdots,X_d, V$ be bounded C^1 vector fields on a C^3 Riemannian manifold $(\mathbf{N},\langle\cdot,\cdot\rangle)$. Let $B_t = (B_t^1,\cdots,B_t^d)$ be a real d-dimensional Brownian motion on a probability space $(\Theta,\mathcal{F},\mathcal{F}_t,\mathbb{Q})$ with generator Δ. An $\mathbf{N}$-valued semimartingale $\mathbf{x} = (\mathbf{x}_t)_{t\in\mathbb{R}_+}$ defined up to a stopping time τ is said to be a solution of the following Stratonovich SDE

$$d\mathbf{x}_t = \sum_{i=1}^{d} X_i(\mathbf{x}_t) \circ dB_t^i + V(\mathbf{x}_t)\ dt \tag{2.28}$$

up to τ if for all $f \in C^\infty(\mathbf{N})$,

$$f(\mathbf{x}_t) = f(\mathbf{x}_0) + \int_0^t \sum_{i=1}^{d} X_i f(\mathbf{x}_s) \circ dB_s^i + \int_0^t V f(\mathbf{x}_s)\ ds,\ 0 \le t < \tau.$$

Call a second order differential operator $\mathbf{A}$ the generator of $\mathbf{x}$ if

$$f(\mathbf{x}_t) - f(\mathbf{x}_0) - \int_0^t \mathbf{A} f(\mathbf{x}_s)\ ds,\ 0 \le t < \tau,$$

is a local martingale for all $f \in C^\infty(\mathbf{N})$. It is known (cf. [**Hs**]) that (2.28) has a unique solution with a Hörmander type second order elliptic operator generator

$$\mathbf{A} = \sum_{i=1}^{d} X_i^2 + V.$$

If $X_1, \cdots, X_d, V$ are such that the corresponding $\mathbf{A}$ is the Laplace operator on $\mathbf{N}$, then the solution of the SDE (2.28) generates the Brownian motion on $\mathbf{N}$. However, there is no general way of obtaining such a collection of vector fields on a general Riemannian manifold.

To obtain the Brownian motion $(\mathbf{x}_t)_{t\in\mathbb{R}_+}$ on $\mathbf{N}$, we adopt the Eells-Elworthy-Malliavin approach (cf. [**El**]) by constructing a canonical diffusion on the frame bundle $O(\mathbf{N})$. Let $TO(\mathbf{N})$ be the tangent space of $O(\mathbf{N})$. For $x \in \mathbf{N}$ and $\mathbf{w} \in O_x(\mathbf{N})$, an element $\mathbf{u} \in \mathbf{T_w O}(\mathbf{N})$ is *vertical* if its projection on $T_x\mathbf{N}$ vanishes. The canonical connection associated with the metric defines a *horizontal* complement, identified with $T_x\mathbf{N}$. For a vector $v \in T_x\mathbf{N}$, $\mathbf{H}_v$, the horizontal lift of v to $T_{\mathbf{w}}O(\mathbf{N})$, describes the infinitesimal parallel transport of the frame $\mathbf{w}$ in the direction of v.

Suppose $\mathbf{N}$ has dimension m. Let $B_t = (B_t^1, \cdots, B_t^m)$ be an m-dimensional Brownian motion on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with generator Δ. Let $\{e_i\}$ be the standard orthonormal basis on $\mathbb{R}^m$. Then, we consider the canonical diffusion on the orthonormal bundle $O(\mathbf{N})$ given by the solution $\mathbf{w}_t$ of the Stratonovich SDE

$$\begin{aligned} d\mathbf{w}_t &= \sum_{i=1}^{m} \mathbf{H}_i(\mathbf{w}_t) \circ dB_t^i, \\ \mathbf{w}_0 &= \mathbf{w}, \end{aligned}$$

where $\mathbf{H}_i(\mathbf{w}_t)$ is the horizontal lift of $\mathbf{w}_t e_i$ to $\mathbf{w}_t$. The Brownian motion $\mathbf{x} = (\mathbf{x}_t)_{t\in\mathbb{R}_+}$ can be obtained as the projection on $\mathbf{N}$ of $\mathbf{w}_t$ for any choice of $\mathbf{w}_0$ which projects to $\mathbf{x}_0$. We can regard $\mathbf{x}(\cdot)$ as a measurable map from Θ to $C_{\mathbf{x}_0}(\mathbb{R}_+, \mathbf{N})$, the space of continuous functions ρ from $\mathbb{R}_+$ to $\mathbf{N}$ with $\rho(0) = \mathbf{x}_0$. So

$$\mathbb{P} := \mathbb{Q}(\mathbf{x}^{-1})$$

gives the probability distribution of the Brownian motion paths in Ω_+. For any $\tau \in \mathbb{R}_+$, let $C_{\mathbf{x}_0}([0,\tau], \mathbf{N})$ denote the space of continuous functions ρ from $[0,\tau]$ to $\mathbf{N}$ with $\rho(0) = \mathbf{x}_0$. Then $\mathbf{x}$ also induces a measurable map $\mathbf{x}_{[0,\tau]}: \Theta \to C_{\mathbf{x}_0}([0,\tau], \mathbf{N})$ which sends $\underline{\omega}$ to $(\mathbf{x}_t(\underline{\omega}))_{t\in[0,\tau]}$. We see that

$$\mathbb{P}_\tau := \mathbb{Q}(\mathbf{x}_{[0,\tau]}^{-1})$$

describes the distribution probability of the Brownian motion paths on $\mathbf{N}$ up to time τ.

More generally, we can obtain in the same way, and on the same probability space, a diffusion with generator $\Delta + V_1$, where V_1 is a bounded C^1 vector field on $\mathbf{N}$. We denote by $\overline{V}_1$ the horizontal lift of V_1 in $O(\mathbf{N})$. Consider the Stratonovich SDE on $O(\mathbf{N})$

$$\begin{aligned} d\mathbf{u}_t &= \sum_{i=1}^m \mathbf{H}_i(\mathbf{u}_t) \circ dB_t^i + \overline{V}_1(\mathbf{u}_t)\ dt, \\ \mathbf{u}_0 &= \mathbf{u}. \end{aligned}$$

Then, the diffusion process $\mathbf{y} = (\mathbf{y}_t)_{t\in\mathbb{R}_+}$ on $\mathbf{N}$ with infinitesimal generator $\Delta_{\mathbf{N}} + V_1$ can be obtained as the projection on $\mathbf{N}$ of the solution $\mathbf{u}_t$ for any choice of $\mathbf{u}_0$ which projects to $\mathbf{y}_0$. We call $\mathbf{u}_t$ the horizontal lift of $\mathbf{y}_t$. Let $\mathbb{P}^1$ be the distribution of $\mathbf{y}$ in $C_{\mathbf{y}_0}(\mathbb{R}_+, \mathbf{N})$ and let $\mathbb{P}^1_\tau$ ($\tau \in \mathbb{R}_+$) be the distribution of $(\mathbf{y}_t(\underline{\omega}))_{t\in[0,\tau]}$ in $C_{\mathbf{y}_0}([0,\tau], \mathbf{N})$, respectively. Then

$$\mathbb{P}^1 = \mathbb{Q}(\mathbf{y}^{-1}), \quad \mathbb{P}^1_\tau = \mathbb{Q}(\mathbf{y}_{[0,\tau]}^{-1}).$$

We now express the relation between $\mathbb{P}^1_\tau$ and $\mathbb{P}_\tau$, as described by the Girsanov-Cameron-Martin formula. Let M^1_t be the random process on $\mathbb{R}$ satisfying $\mathrm{M}^1_0 = 1$ and the Stratonovich SDE

$$d\mathrm{M}^1_t = \mathrm{M}^1_t \langle \frac{1}{2} V_1(\mathbf{x}_t), \mathbf{w}_t \circ dB_t \rangle_{\mathbf{x}_t} - \mathrm{M}^1_t \left(\| \frac{1}{2} V_1(\mathbf{x}_t) \|^2 + \mathrm{Div} \left(\frac{1}{2} V_1(\mathbf{x}_t) \right) \right).$$

Then

$$\mathrm{M}^1_t = \exp\left\{ \int_0^t \langle \frac{1}{2} V_1(\mathbf{x}_s(\underline{\omega})), \mathbf{w}_s(\underline{\omega}) \circ dB_s(\underline{\omega}) \rangle_{\mathbf{x}_s} - \int_0^t \left(\| \frac{1}{2} V_1(\mathbf{x}_s(\underline{\omega})) \|^2 + \mathrm{Div}\big(\frac{1}{2} V_1(\mathbf{x}_s(\underline{\omega})) \big) \right)\ ds \right\}.$$

In the more familiar Ito's stochastic integral form, we have

$$d\mathrm{M}^1_t = \frac{1}{2} \mathrm{M}^1_t \langle V_1(\mathbf{x}_t), \mathbf{w}_t dB_t \rangle_{\mathbf{x}_t}$$

and

$$\mathrm{M}^1_t = \exp\left\{ \frac{1}{2} \int_0^t \langle V_1(\mathbf{x}_s(\underline{\omega})), \mathbf{w}_s(\underline{\omega}) dB_s(\underline{\omega}) \rangle_{\mathbf{x}_s} - \frac{1}{4} \int_0^t \| V_1(\mathbf{x}_s(\underline{\omega})) \|^2\ ds \right\}. \tag{2.29}$$

Since each $\mathbb{E}_{\mathbb{Q}} \left(\exp\{ \frac{1}{4} \int_0^t \| V_1(\mathbf{x}_s(\underline{\omega})) \|^2\ ds \} \right)$ is finite, we have by Novikov ([**N**]), that $\mathrm{M}^1_t, t \geq 0$, is a continuous $(\mathcal{F}_t)$-martingale, i.e.,

$$\mathbb{E}_{\mathbb{Q}} \left(\mathrm{M}^1_t \right) = 1 \ \text{ for every } t \geq 0,$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation of a random variable with respect to $\mathbb{Q}$. For $\tau \in \mathbb{R}_+$, let $\mathbb{Q}_\tau^1$ be the probability on Θ, which is absolutely continuous with respect to $\mathbb{Q}$ with

$$\frac{d\mathbb{Q}_\tau^1}{d\mathbb{Q}}(\underline{\omega}) = \mathrm{M}_\tau^1(\underline{\omega}).$$

Note that M_τ^1 is a martingale, so that the projection of $\mathbb{Q}_\tau^1$ on the coordinates up to $\tau' < \tau$ is given by the same formula. A version of the Girsanov theorem (cf. [**El**, Theorem 11B]) says that $((\mathbf{y}_t)_{t\in[0,\tau]}, \mathbb{Q})$ is isonomous to $((\mathbf{x}_t)_{t\in[0,\tau]}, \mathbb{Q}_\tau^1)$ in the sense that for any finite numbers $\tau_1, \cdots, \tau_s \in [0,\tau]$,

$$\left(\mathbb{Q}(\mathbf{y}_{\tau_1}^{-1}), \cdots, \mathbb{Q}(\mathbf{y}_{\tau_s}^{-1})\right) = \left(\mathbb{Q}_\tau^1(\mathbf{x}_{\tau_1}^{-1}), \cdots, \mathbb{Q}_\tau^1(\mathbf{x}_{\tau_s}^{-1})\right). \tag{2.30}$$

(The coefficients in (2.29) differ from the ones in [**El**] because B_t has generator Δ.) Let $\mathbb{Q}^1$ be the probability on Θ associated with $\{\mathbb{Q}_\tau^1\}_{\tau\in\mathbb{R}_+}$. Then (2.30) intuitively means that by changing the measure $\mathbb{Q}$ on Θ to $\mathbb{Q}^1$, $\mathbf{x}$ has the same distribution as $(\mathbf{y}, \mathbb{Q})$. As a consequence, we have $\mathbb{P}_\tau^1 = \mathbb{Q}_\tau^1(\mathbf{x}^{-1})$ for all $\tau \in \mathbb{R}_+$ and hence

$$\frac{d\mathbb{P}_\tau^1}{d\mathbb{P}_\tau}\left(\mathbf{x}_{[0,\tau]}\right) = \mathbb{E}_{\mathbb{Q}}\left(\mathrm{M}_\tau^1 \middle| \mathcal{F}(\mathbf{x}_{[0,\tau]})\right), \text{ a.s.},$$

where $\mathbb{E}_{\mathbb{Q}}(\cdot \mid \cdot)$ is the conditional expectation with respect to $\mathbb{Q}$ and $\mathcal{F}(\mathbf{x}_{[0,\tau]})$ is the smallest σ-algebra on Θ for which the map $\mathbf{x}_{[0,\tau]}$ is measurable.

Let V_2 be another bounded C^1 vector field on $\mathbf{N}$. Consider the diffusion process $\mathbf{z} = (\mathbf{z}_t)_{t\in\mathbb{R}_+}$ on $\mathbf{N}$ with the same initial point as $\mathbf{y}$, but with infinitesimal generator $\Delta_{\mathbf{N}} + V_1 + V_2$. Let $\mathbb{P}^2$ be the distribution of $\mathbf{z}$ in the space of continuous paths on $\mathbf{N}$ and let $\mathbb{P}_\tau^2(\tau \in \mathbb{R}_+)$ be the distribution of $(\mathbf{z}_t(\underline{\omega}))_{t\in[0,\tau]}$. The Girsanov-Cameron-Martin formula on manifolds (cf. [**El**, Theorem 11C]) says that $\mathbb{P}_\tau^2$ is absolutely continuous with respect to $\mathbb{P}_\tau^1$ with

$$\frac{d\mathbb{P}_\tau^2}{d\mathbb{P}_\tau^1}(\mathbf{y}_{[0,\tau]}) = \mathbb{E}_{\mathbb{Q}}\left(\mathrm{M}_\tau^2 \middle| \mathcal{F}(\mathbf{y}_{[0,\tau]})\right), \text{ a.s.}, \tag{2.31}$$

where

$$\mathrm{M}_\tau^2(\underline{\omega}) = \exp\left\{\frac{1}{2}\int_0^\tau \langle V_2(\mathbf{y}_s(\underline{\omega})), \mathbf{u}_s(\underline{\omega})dB_s(\underline{\omega})\rangle_{\mathbf{y}_s} - \frac{1}{4}\int_0^\tau \|V_2(\mathbf{y}_s(\underline{\omega}))\|^2\, ds\right\}$$

and $\mathcal{F}(\mathbf{y}_{[0,\tau]})$ is the smallest σ-algebra on Θ for which the map $\mathbf{y}_{[0,\tau]}$ is measurable.

3. Regularity of the linear drift and the stochastic entropy for $\Delta + Y$

Consider a one-parameter family of variations $\{\mathcal{L}^\lambda = \Delta + Y + Z^\lambda : \ |\lambda| < 1\}$ of $\mathcal{L}$ with $Z^0 = 0$ and Z^λ twice differentiable in λ so that $\sup_{\lambda\in(-1,1)} \max\{\|\frac{dZ^\lambda}{d\lambda}\|, \|\frac{d^2Z^\lambda}{d\lambda^2}\|\}$ is finite. Assume each $\mathcal{L}^\lambda$ is subordinate to the stable foliation, $Y + Z^\lambda$ is such that $(Y + Z^\lambda)^*$, the dual of $(Y + Z^\lambda)$ in the cotangent bundle to the stable foliation over SM, satisfies $d(Y + Z^\lambda)^* = 0$ leafwisely and $\mathrm{pr}(-\langle \overline{X}, Y + Z^\lambda\rangle) > 0$. Then each $\mathcal{L}^\lambda$ has a unique harmonic measure. Hence the linear drift for $\mathcal{L}^\lambda$, denoted $\ell_\lambda := \ell_{\mathcal{L}^\lambda}$, and the stochastic entropy for

$\mathcal{L}^\lambda$, denoted $\overline{h}_\lambda := h_{\mathcal{L}^\lambda}$, are well-defined. In this section, we show the differentiability of $\overline{\ell}_\lambda$ and $\overline{h}_\lambda$ in λ at 0 (Theorem 3.3 and Theorem 3.9).

Consider the diffusion process of the stable foliation of $S\widetilde{M}$ corresponding to $\mathcal{L}^\lambda$ ($\lambda \in (-1,1)$). Let $B_t = (B_t^1, \cdots, B_t^m)$ be an m-dimensional Brownian motion on a probability space $(\Theta, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ with generator Δ. For each $\mathbf{v} = (x, \xi) \in S\widetilde{M}$, $W^s(\mathbf{v})$ can be identified with $\widetilde{M} \times \{\xi\}$, or simply $\widetilde{M}$. So for each $\lambda \in (-1,1)$, there exists the diffusion process $\mathbf{y}_{\mathbf{v}}^\lambda = (\mathbf{y}_{\mathbf{v},t}^\lambda)_{t\in\mathbb{R}_+}$ on $W^s(\mathbf{v})$ starting from $\mathbf{v}$ with infinitesimal generator $\mathcal{L}_{\mathbf{v}}^\lambda$. Each $\mathbf{y}_{\mathbf{v}}^\lambda$ induces a measurable map from Θ to $C_{\mathbf{v}}(\mathbb{R}_+, W^s(\mathbf{v})) \subset \Omega_+$ and $\overline{\mathbb{P}}_{\mathbf{v}}^\lambda := \mathbb{Q}((\mathbf{y}_{\mathbf{v}}^\lambda)^{-1})$ gives the distribution probability of $\mathbf{y}_{\mathbf{v}}^\lambda$ in $C_{\mathbf{v}}(\mathbb{R}_+, W^s(\mathbf{v}))$. For any $\tau \in \mathbb{R}_+$, let $\overline{\mathbb{P}}_{\mathbf{v},\tau}^\lambda$ be the distribution of $(\mathbf{y}_{\mathbf{v},t}^\lambda)_{t\in[0,\tau]}$ in $C_{\mathbf{v}}([0,\tau], W^s(\mathbf{v}))$. We have by the Girsanov-Cameron-Martin formula on manifolds (2.31) that $\overline{\mathbb{P}}_{\mathbf{v},\tau}^\lambda$ is absolutely continuous with respect to $\overline{\mathbb{P}}_{\mathbf{v},\tau}^0$ with

$$\frac{d\overline{\mathbb{P}}_{\mathbf{v},\tau}^\lambda}{d\overline{\mathbb{P}}_{\mathbf{v},\tau}^0}(\mathbf{y}_{\mathbf{v},[\mathbf{0},\tau]}^0) = \mathbb{E}_{\mathbb{Q}}\left(\overline{\mathrm{M}}_\tau^\lambda | \mathcal{F}(\mathbf{y}_{\mathbf{v},[\mathbf{0},\tau]}^0)\right), \quad \text{a.s.}, \tag{3.1}$$

where

$$\overline{\mathrm{M}}_\tau^\lambda(\underline{\omega}) = \exp\left\{\frac{1}{2}\int_0^\tau \langle Z^\lambda(\mathbf{y}_{\mathbf{v},s}^0(\underline{\omega})), \mathbf{u}_{\mathbf{v},s}^0(\underline{\omega}) dB_s(\underline{\omega})\rangle_{\mathbf{y}_{\mathbf{v},s}^0} - \frac{1}{4}\int_0^\tau \|Z^\lambda(\mathbf{y}_{\mathbf{v},s}^0(\underline{\omega}))\|^2\, ds\right\},$$

$\mathbf{u}_{\mathbf{v},t}^0$ is the horizontal lift of $\mathbf{y}_{\mathbf{v},t}^0$ to $O(W^s(\mathbf{v}))$ and $\mathcal{F}(\mathbf{y}_{\mathbf{v},[\mathbf{0},\tau]}^0)$ is the smallest σ-algebra on Θ for which the map $\mathbf{y}_{\mathbf{v},[\mathbf{0},\tau]}^0$ is measurable.

For each $\lambda \in (-1,1)$, let $\mathbf{m}^\lambda$ be the unique $\mathcal{L}^\lambda$-harmonic measure and $\widetilde{\mathbf{m}}^\lambda$ be its G-invariant extension to $S\widetilde{M}$. We see that $\overline{\mathbb{P}}^\lambda = \int \overline{\mathbb{P}}_{\mathbf{v}}^\lambda\, d\widetilde{\mathbf{m}}^\lambda(\mathbf{v})$ is the shift invariant measure on $\widetilde{\Omega}_+$ corresponding to $\widetilde{\mathbf{m}}^\lambda$ and we restrict $\overline{\mathbb{P}}^\lambda$ to Ω_+. Consider the space $\overline{\Theta} = SM \times \Theta$ with product σ-algebra and probability $\overline{\mathbb{Q}}^\lambda$, $d\overline{\mathbb{Q}}^\lambda(\mathbf{v}, \underline{\omega}) = d\mathbb{Q}(\underline{\omega}) \times d\mathbf{m}^\lambda(\mathbf{v})$. Let $\mathbf{y}_t^\lambda :\ SM \times \Theta \to S\widetilde{M}$ be such that

$$\mathbf{y}_t^\lambda(\mathbf{v}, \underline{\omega}) = \mathbf{y}_{\mathbf{v},t}^\lambda(\underline{\omega}), \text{ for } (\mathbf{v}, \underline{\omega}) \in SM \times \Theta.$$

Then $\mathbf{y}^\lambda = (\mathbf{y}_t^\lambda)_{t\in\mathbb{R}_+}$ defines a random process on the probability space $(\overline{\Theta}, \overline{\mathbb{Q}}^\lambda)$ with images in the space of continuous paths on the stable leaves of $S\widetilde{M}$.

Simply write $\mathbf{y}_t = \mathbf{y}_t^0$ and let $\mathbf{u}_t$ be such that $\mathbf{u}_t(\mathbf{v}, \underline{\omega}) = \mathbf{u}_{\mathbf{v},t}^0(\underline{\omega})$ for $(\mathbf{v}, \underline{\omega}) \in \overline{\Theta}$. Denote by $(Z^\lambda)_0' := (dZ^\lambda/d\lambda)|_{\lambda=0}$. We consider three random variables on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$:

$$\begin{aligned}
\mathbf{M}_t^0 &:= \frac{1}{2}\int_0^t \langle (Z^\lambda)_0'(\mathbf{y}_s), \mathbf{u}_s dB_s\rangle_{\mathbf{y}_s}, \\
\mathbf{Z}_{\ell,t}^0 &:= [d_{\mathcal{W}}(\mathbf{y}_0, \mathbf{y}_t) - t\ell_{\mathcal{L}^0}], \\
\mathbf{Z}_{h,t}^0 &:= -\left[\mathbf{1}_{\{d(\mathbf{y}_0,\mathbf{y}_t)\geq 1\}} \cdot \ln \mathbf{G}(\mathbf{y}_0, \mathbf{y}_t) + t h_{\mathcal{L}^0}\right],
\end{aligned}$$

where $\mathbf{1}_B$ is the characteristic function of the event B. We will prove the following two Propositions separately in Sections 3.1 and 3.2.

Proposition 3.1. *The laws of the random vectors* $(\mathbf{Z}^0_{\ell,t}/\sqrt{t}, \mathbf{M}^0_t/\sqrt{t})$ *under* $\overline{\mathbb{Q}}^0$ *converge in distribution as* t *tends to* $+\infty$ *to a bivariate centered normal law with some covariance matrix* Σ_ℓ*. The covariance matrices of* $(\mathbf{Z}^0_{\ell,t}/\sqrt{t}, \mathbf{M}^0_t/\sqrt{t})$ *under* $\overline{\mathbb{Q}}^0$ *converge to* Σ_ℓ*.*

Proposition 3.2. *The laws of the random vectors* $(\mathbf{Z}^0_{h,t}/\sqrt{t}, \mathbf{M}^0_t/\sqrt{t})$ *under* $\overline{\mathbb{Q}}^0$ *converge in distribution as* t *tends to* $+\infty$ *to a bivariate centered normal law with some covariance matrix* Σ_h*. The covariance matrices of* $(\mathbf{Z}^0_{h,t}/\sqrt{t}, \mathbf{M}^0_t/\sqrt{t})$ *under* $\overline{\mathbb{Q}}^0$ *converge to* Σ_h*.*

3.1. **The differential of the linear drift.** For any $\lambda \in (-1, 1)$, let $\overline{\ell}_\lambda$ be the linear drift of $\mathcal{L}^\lambda$. The main result of this subsection is the following.

Theorem 3.3. *The function* $\lambda \mapsto \overline{\ell}_\lambda$ *is differentiable at* 0 *with*

$$\frac{d\overline{\ell}_\lambda}{d\lambda}\Big|_{\lambda=0} = \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{\ell,t}\mathbf{M}^0_t).$$

We fix a fundamental domain M_0 of $\widetilde{M}$ and identify Ω_+ with the lift of its elements in $\widetilde{\Omega}_+$ starting from M_0. In the following two subsections, we restrict the probabilities on $\widetilde{\Omega}_+$ to Ω_+. For any $\tau \in \mathbb{R}_+$, recall that $\overline{\mathbb{P}}^\lambda_{\mathbf{v},\tau}$ is the distribution of $(\mathbf{y}^\lambda_{\mathbf{v},t})_{t\in[0,\tau]}$ in $C_{\mathbf{v}}([0,\tau], W^s(\mathbf{v}))$. By an abuse of notation, we can also regard $\overline{\mathbb{P}}^\lambda_{\mathbf{v},\tau}$ as a measure on Ω_+ whose value only depends on $(\omega(t))_{t\in[0,\tau]}$ for $\omega = (\omega(t))_{t\in\mathbb{R}_+} \in \Omega_+$. Let $\overline{\mathbb{P}}^\lambda_t = \int \overline{\mathbb{P}}^\lambda_{\mathbf{v},t}\, d\mathbf{m}^\lambda(\mathbf{v})$. Then

$$\overline{\ell}_\lambda = \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}_{\overline{\mathbb{P}}^\lambda_t}\left(d_{\mathcal{W}}(\omega(0),\omega(t))\right).$$

We will prove Theorem 3.3 in two steps. Firstly, using negative curvature, we find a finite number D_ℓ such that for all $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is from Lemma 3.4) and all $t > 0$,

$$\left|\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d_{\mathcal{W}}(\omega(0),\omega(t))\right) - t\overline{\ell}_\lambda\right| \le D_\ell. \tag{3.2}$$

In particular, for $t = \lambda^{-2}$,

$$\left|\lambda\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d_{\mathcal{W}}(\omega(0),\omega(\lambda^{-2}))\right) - \frac{1}{\lambda}\overline{\ell}_\lambda\right| \le \lambda D_\ell.$$

Thanks to (3.2), the study of $\frac{d\overline{\ell}_\lambda}{d\lambda}\Big|_{\lambda=0} = \lim_{\lambda\to0}\frac{1}{\lambda}\left|\overline{\ell}_\lambda - \overline{\ell}_0\right|$ reduces to the study of

$$\lim_{\lambda\to0}\left|\lambda\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d_{\mathcal{W}}(\omega(0),\omega(\lambda^{-2}))\right) - \frac{1}{\lambda}\overline{\ell}_0\right|.$$

Setting $\lambda = \pm1/\sqrt{t}$, the second step is to show

$$\lim_{t\to+\infty}(\mathrm{I})^t_\ell = \lim_{t\to+\infty}\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{\sqrt{t}}\mathbf{Z}^0_{\ell,t}\cdot\overline{\mathrm{M}}^\lambda_t\right), \tag{3.3}$$

where

$$(\mathrm{I})^t_\ell := \mathbb{E}_{\overline{\mathbb{P}}^\lambda_t}\left(\frac{1}{\sqrt{t}}\left(d_{\mathcal{W}}(\omega(0),\omega(t)) - t\overline{\ell}_0\right)\right).$$

Using notations from the previous subsection, we know $d\overline{\mathbb{P}}_t^\lambda/d\overline{\mathbb{P}}_t^0$ is given by $\overline{\mathrm{M}}_t^\lambda$. Each $\overline{\mathbb{P}}_t^\lambda$ is a random perturbation of the distribution $\overline{\mathbb{P}}_t^0$ on the path space and at the scale of $\lambda = 1/\sqrt{t}$, $\overline{\mathrm{M}}_t^\lambda$ converge in distribution to $e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}$ as t goes to infinity. Consequently, $\frac{1}{\sqrt{t}}\mathbf{Z}^0_{\ell,t} \cdot \overline{\mathrm{M}}_t^\lambda$ converge in distribution to $\mathbf{Z}^0_\ell e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}$, which can be identified with $\lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{\ell,t}\mathbf{M}^0_t)$ using Proposition 3.1 (see Lemma 3.8).

We therefore follow the above discussion and prove (3.2) and (3.3). Let us first show that there is a finite number D_ℓ such that for all $\lambda \in [-\delta_1, \delta_1]$ and all $t > 0$,

$$\left|\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d_{\mathcal{W}}(\omega(0), \omega(t))\right) - t\bar{\ell}_\lambda\right| \le D_\ell.$$

Since the $\mathcal{L}^\lambda$-diffusion has leafwise infinitesimal generator $\mathcal{L}^\lambda_{\mathbf{v}}$ and $\mathbb{P}^\lambda$ is stationary, we have

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(b_{\omega(0)}(\omega(t))\right) &= \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\int_0^t \frac{\partial}{\partial s} b_{\omega(0)}(\omega(s))\, ds\right)\\
&= \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\int_0^t (\mathcal{L}^\lambda_{\omega(0)} b_{\omega(0)})(\omega(s))\, ds\right)\\
&= t\int_{M_0 \times \partial\widetilde{M}} \mathcal{L}^\lambda_{\mathbf{v}} b_{\mathbf{v}}\, d\widetilde{\mathbf{m}}^\lambda\\
&= t\bar{\ell}_\lambda.
\end{aligned}$$

So, proving (3.2) reduces to showing that for all $\lambda \in [-\delta_1, \delta_1]$ and all $t > 0$,

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\left|d_{\mathcal{W}}(\omega(0), \omega(t)) - b_{\omega(0)}(\omega(t))\right|\right) < D_\ell, \tag{3.4}$$

which intuitively means that for all $\lambda, \mathbf{v}$, the $\mathcal{L}^\lambda_{\mathbf{v}}$-diffusion orbits $\omega(t)$ does not accumulates to the point $\xi \in \partial\widetilde{M}$ such that $\omega(0) = \mathbf{v} = (x, \xi)$. For $\omega \in \Omega_+$, we still denote ω its projection to $\widetilde{M}$. Then the leafwise distance $d_{\mathcal{W}}(\omega(0), \omega(t))$ in (3.4) is just $d(\omega(0), \omega(t))$.

We first take a look at the distribution of $\omega(\infty) := \lim_{s\to+\infty}\omega(s)$ on the boundary. Let $x \in \widetilde{M}$ be a reference point and let $\iota > 0$ be a positive number. Define

$$d_x^\iota(\zeta, \eta) := \exp\left(-\iota(\zeta|\eta)_x\right),\ \forall \zeta, \eta \in \partial\widetilde{M},$$

where $(\zeta|\eta)_x$ is defined as in (2.16). If ι_0 is small, each $d_x^\iota(\cdot,\cdot)$ $(x \in \widetilde{M}, \iota \in (0, \iota_0))$ defines a distance on $\partial\widetilde{M}$ ([**GH**]), the so-called *ι-Busemann distance*, which is related to the Busemann functions since

$$b_{\mathbf{v}}(y) = \lim_{\zeta,\eta\to\xi}\left((\zeta|\eta)_y - (\zeta|\eta)_x\right),\ \text{for any } \mathbf{v} = (x, \xi) \in S\widetilde{M}, y \in \widetilde{M}.$$

The following shadow lemma ([**Moh**, Lemma 2.14], see also [**PPS**]) says that the $\mathcal{L}^\lambda$-harmonic measure has a positive dimension on the boundary in a uniform way.

Lemma 3.4. *There are $D_1, \delta_1, \alpha_1, \iota_1 > 0$ such that for all $\lambda \in [-\delta_1, \delta_1]$, all $\mathbf{v} \in SM$ and all $\zeta \in \partial\widetilde{M}$, $t > 0$,*

$$\overline{\mathbb{P}}^\lambda_{\mathbf{v}}\left(d_x^{\iota_1}(\zeta, \omega(\infty)) \le e^{-t}\right) \le D_1 e^{-\alpha_1 t},$$

where we identify $\omega(s)$ with its projection on $\widetilde{M}$.

As a consequence, we see that for $\overline{\mathbb{P}}^\lambda$-almost all orbits $\omega \in \Omega_+$, the distance between $\omega(s)$ and $\gamma_{\omega(0),\omega(\infty)}$, the geodesic connecting $\omega(0)$ and $\omega(\infty)$, is bounded in the following sense.

Lemma 3.5. *There exists a finite number D_2 such that for all $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is as in Lemma 3.4) and $s \in \mathbb{R}_+$,*

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d(\omega(s), \gamma_{\omega(0),\omega(\infty)})\right) < D_2.$$

Proof. Extend $\overline{\mathbb{P}}^\lambda$ to a shift invariant probability measure $\breve{\mathbb{P}}^\lambda$ on the set of trajectories from $\mathbb{R}$ to SM, by

$$\breve{\mathbb{P}}^\lambda = \int_{SM} \overline{\mathbb{P}}^\lambda_{\mathbf{v}} \otimes (\overline{\mathbb{P}}')^\lambda_{\mathbf{v}}\ d\mathbf{m}^\lambda(\mathbf{v}),$$

where $(\overline{\mathbb{P}}')^\lambda_{\mathbf{v}}$ is the probability describing the reversed $\mathcal{L}^\lambda_{\mathbf{v}}$-diffusion starting from $\mathbf{v}$. Then we have by invariance of $\breve{\mathbb{P}}^\lambda$ that

$$\begin{aligned} \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d(\omega(s), \gamma_{\omega(0),\omega(\infty)})\right) &= \mathbb{E}_{\breve{\mathbb{P}}^\lambda}\left(d(\omega(0), \gamma_{\omega(-s),\omega(\infty)})\right) \\ &= \int \left(d(x, \gamma_{\omega(-s),\omega(\infty)})\right) d\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(\widetilde{\omega}) d(\overline{\mathbb{P}}')^\lambda_{\mathbf{v}}(\widetilde{\omega}(-s)) d\mathbf{m}^\lambda(\mathbf{v}). \end{aligned} \tag{3.5}$$

Fix $\omega(-s) = z$ at distance D from x, and let $\zeta \in \partial\widetilde{M}$ be $\lim_{t\to+\infty} \gamma_{x,z}(t)$. We estimate

$$\int d(x, \gamma_{z,\omega(\infty)})\ d\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(\widetilde{\omega}) = \int_0^{+\infty} \overline{\mathbb{P}}^\lambda_{\mathbf{v}}(d(x, \gamma_{z,\omega(\infty)}) > t)\ dt.$$

For $t \geq D$, it is clear that $\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(d(x, \gamma_{z,\omega(\infty)}) > t) = 0$. For $t < D$, if $d(x, \gamma_{z,\omega(\infty)}) > t$, then $d_x^{\iota_1}(\zeta, \omega(\infty)) \leq Ce^{-\iota_1 t}$ for some constant C and hence we have by Lemma 3.4 that

$$\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(d(x, \gamma_{z,\omega(\infty)}) > t) \leq CD_1 e^{-\alpha_1\iota_1 t}.$$

Therefore,

$$\int d(x, \gamma_{z,\omega(\infty)})\ d\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(\widetilde{\omega}) \leq \int_1^D CD_1 e^{-\alpha_1\iota_1 t}\ dt + 1 \leq \frac{CD_1}{\alpha_1\iota_1} e^{-\alpha_1\iota_1} + 1 := D_2.$$

Using (3.5), we obtain that $\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d(\omega(s), \gamma_{\omega(0),\omega(\infty)})\right)$ is bounded by D_2 as well. $\square$

Now, using Lemmas 3.4 and 3.5, we prove in Lemma 3.6 that there is a bounded square integrable difference between $d_{\mathcal{W}}(\omega(0), \omega(s))$ and $b_{\omega(0)}(\omega(s))$ for all s (cf. [**Ma**, Lemma 3.4]). This Lemma 3.6 implies (3.4) and therefore concludes the proof of (3.2).

Lemma 3.6. *There exists a finite number D_3 such that for all $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is as in Lemma 3.4) and $s \in \mathbb{R}_+$,*

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\left|d_{\mathcal{W}}(\omega(0), \omega(s)) - b_{\omega(0)}(\omega(s))\right|^2\right) < D_3.$$

Proof. It is clear that

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\left|d_{\mathcal{W}}(\omega(0),\omega(s)) - b_{\omega(0)}(\omega(s))\right|^2\right) = 4\int \left(\omega(s)\middle|\xi\right)_x^2 \ d\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(\omega)d\mathbf{m}^\lambda(\mathbf{v}),$$

where $\omega(0) = \mathbf{v} = (x,\xi)$ and $\left(\omega(s)\middle|\xi\right)_x := \lim_{y\to\xi}\left(\omega(s)\middle|y\right)_x$ (see (2.16) for the definition of $(z|y)_x$ for $x,y,z \in \widetilde{M}$). So, it suffices to estimate

$$\int_0^{+\infty} \overline{\mathbb{P}}^\lambda_{\mathbf{v}}((\omega(s)\big|\xi)_x^2 > t)\ dt = \int_0^{+\infty} \overline{\mathbb{P}}^\lambda_{\mathbf{v}}((\omega(s)\big|\xi)_x > \sqrt{t})\ dt.$$

For each $t > 0$, divide the event $\{\omega \in \Omega_+ :\ (\omega(s)\big|\xi)_x > \sqrt{t}\}$ into two sub-events

$$\begin{aligned} A_1(t) &:= \{\omega \in \Omega_+ :\ (\omega(s)\big|\xi)_x > \sqrt{t},\ (\omega(s)\big|\omega(\infty))_x \geq \frac{1}{4}\sqrt{t}\}, \\ A_2(t) &:= \{\omega \in \Omega_+ :\ (\omega(s)\big|\xi)_x > \sqrt{t},\ (\omega(s)\big|\omega(\infty))_x < \frac{1}{4}\sqrt{t}\}. \end{aligned}$$

We estimate $\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(A_i(t))$, $i = 1,2$, successively. Since M is a closed connected negatively curved Riemannian manifold, its universal cover $\widetilde{M}$ is Gromov hyperbolic in the sense that there exists $\delta > 0$ such that for any $x_1, x_2, x_3 \in \widetilde{M}$,

$$(x_1|x_2)_x \geq \min\{(x_1|x_3)_x,\ (x_2|x_3)_x\} - \delta.$$

So on each $A_1(t)$, where $t > 64\delta^2$, we have

$$(\xi|\omega(\infty))_x \geq \frac{1}{8}\sqrt{t}.$$

Hence, by Lemma 3.4,

$$\overline{\mathbb{P}}^\lambda_{\mathbf{v}}(A_1(t)) \leq \overline{\mathbb{P}}^\lambda_{\mathbf{v}}((\xi|\omega(\infty))_x \geq \frac{1}{8}\sqrt{t}) = \overline{\mathbb{P}}^\lambda_{\mathbf{v}}(d_x^{\iota_1}(\omega(\infty),\xi) < e^{-\frac{1}{8}\iota_1\sqrt{t}}) \leq D_1 e^{-\frac{1}{8}\iota_1\alpha_1\sqrt{t}},$$

where the last quantity is integrable with respect to t, independent of s. For $\omega \in A_2(t)$,

$$d_{\mathcal{W}}(\omega(0),\omega(s)) \geq (\omega(s)\big|\xi)_x > \sqrt{t}.$$

On the other hand, the point $y(s)$ on $\gamma_{\omega(0),\omega(\infty)}$ closest to $\omega(s)$ satisfies

$$(\omega(s)\big|y(s))_x \leq (\omega(s)\big|\omega(\infty))_x < \frac{1}{4}\sqrt{t}.$$

So we must have

$$d(\omega(s),\gamma_{\omega(0),\omega(\infty)}) > \frac{1}{2}\sqrt{t}.$$

Hence,

$$\int_0^\infty \overline{\mathbb{P}}^\lambda_{\mathbf{v}}(A_2(t))\ dt \leq \int \overline{\mathbb{P}}^\lambda_{\mathbf{v}}\left(d(\omega(s),\gamma_{\omega(0),\omega(\infty)}) > \frac{1}{2}\sqrt{t}\right)\ dt\ d\mathbf{m}^\lambda(\mathbf{v}),$$

which, by the same argument as the one used in the proof of Lemma 3.5, is bounded from above by some constant independent of s. □

To show $\lim_{t\to+\infty}(\mathrm{I})_\ell^t = \lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{\ell,t}\mathbf{M}^0_t)$, we first prove Proposition 3.1.

Proof of Proposition 3.1. Let $(\mathbf{Z}_t^0)_{t\in\mathbb{R}_+}$, u_0 be as in Proposition 2.18. The process $(\mathbf{Z}_t^0)_{t\in\mathbb{R}_+}$ is a centered martingale with stationary increments and its law under $\overline{\mathbb{P}}^0$ is the same as the law of $(\overline{\mathbf{Z}}_t^0)_{t\in\mathbb{R}_+}$ under $\overline{\mathbb{Q}}^0$, where $(\overline{\mathbf{Z}}_t^0)_{t\in\mathbb{R}_+}$ on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ is given by

$$\overline{\mathbf{Z}}_t^0(\mathbf{v}, \underline{\omega}) = -b_{\mathbf{v}}\left(\mathbf{y}_{\mathbf{v},t}(\underline{\omega})\right) + t\overline{\ell}_0 + u_0(\mathbf{y}_{\mathbf{v},t}(\underline{\omega}))) - u_0(\mathbf{v}).$$

The pair $(-\overline{\mathbf{Z}}_t^0, \mathbf{M}_t^0)$ is a centered martingale on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ with stationary increments. To show $(-\overline{\mathbf{Z}}_t^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ converge in distribution to a bivariate centered normal vector, it suffices to show for any $(a,b)\in\mathbb{R}^2$, the combination $-a\overline{\mathbf{Z}}_t^0/\sqrt{t} + b\mathbf{M}_t^0/\sqrt{t}$ converge to a centered normal distribution. The martingales $\overline{\mathbf{Z}}_t^0$ and $\mathbf{M}_t^0$ on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ have integrable increasing process functions $2\|\overline{X} + \nabla u_0\|^2$ and $\|(Z^\lambda)_0'\|^2$, respectively. Using Schwarz inequality, we conclude that $-a\overline{\mathbf{Z}}_t^0 + b\mathbf{M}_t^0$ also has an integrable increasing process function. By Lemma 2.19, $-a\overline{\mathbf{Z}}_t^0/\sqrt{t} + b\mathbf{M}_t^0/\sqrt{t}$ converge in distribution in $\overline{\mathbb{Q}}^0$ to a centered normal law with variance $\Sigma_\ell[a,b] = (a,b)\Sigma_\ell(a,b)^T$ for some matrix Σ_ℓ. Since both $\overline{\mathbf{Z}}_t^0$ and $\mathbf{M}_t^0$ have stationary increments, we also have

$$\Sigma_\ell[a,b] = \frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}^0}\left[(-a\overline{\mathbf{Z}}_t^0 + b\mathbf{M}_t^0)^2\right], \text{ for all } t\in\mathbb{R}_+.$$

The condition (2.27) in Lemma 2.19 is satisfied since the increasing process $\langle -a\overline{\mathbf{Z}}_\cdot^0 + b\mathbf{M}_\cdot^0, -a\overline{\mathbf{Z}}_\cdot^0 + b\mathbf{M}_\cdot^0\rangle_n$ is a Birkhoff sum of a square integrable function over a mixing system (Proposition 2.2). This shows Proposition 3.1 for the pair $(-\overline{\mathbf{Z}}_t^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ instead of the pair $(\mathbf{Z}_{\ell,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$.

Recall that $\overline{\mathbb{P}}_{\mathbf{v}}^0$-a.e. $\omega\in\Omega_+$ is such that $b_{\mathbf{v}}(\omega(t)) - d_{\mathcal{W}}(\omega(0), \omega(t))$ converges to a finite number. Moreover, we have by Lemma 3.6 that

$$\sup_t \mathbb{E}_{\overline{\mathbb{P}}^\lambda}(\left|\mathbf{Z}_{\ell,t}^0 + \overline{\mathbf{Z}}_t^0\right|^2) < +\infty$$

and hence

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}(\frac{1}{t}\left|\mathbf{Z}_{\ell,t}^0 + \overline{\mathbf{Z}}_t^0\right|^2) \to 0, \text{ as t} \to +\infty.$$

Consequently, $(\mathbf{Z}_{\ell,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ has the same limit normal law as $(-\overline{\mathbf{Z}}_t^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ and its covariance matrix under $\overline{\mathbb{Q}}^0$ converges to Σ_ℓ as t goes to infinity. □

We state one more lemma from [**Bi**] on the limit of the expectations of a class of random variables on a common probability space which converge in distribution.

Lemma 3.7. *(cf.* [**Bi**, Theorem 25.12]*) If the random variables X_t $(t\in\mathbb{R})$ on a common probability space converge to X in distribution, and there exists some $q>1$ such that* $\sup_t \mathbb{E}_\nu\left(|X_t|^q\right) < +\infty$*, then X is integrable and*

$$\lim_{t\to+\infty}\mathbb{E}_\nu\left(X_t\right) = \mathbb{E}_\nu\left(X\right).$$

By the above discussion, Theorem 3.3 follows from

Lemma 3.8. *We have* $\lim_{t\to+\infty}(\mathrm{I})_\ell^t = \lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{\ell,t}\mathbf{M}^0_t)$.

Proof. Let $\mathbf{y} = (\mathbf{y}_t)_{t\in\mathbb{R}_+} = (\mathbf{y}_{\mathbf{v},t})_{\mathbf{v}\in SM, t\in\mathbb{R}_+}$ be the diffusion process on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ corresponding to $\mathcal{L}^0$. We know from Section 2.6 that $\overline{\mathbb{P}}^\lambda_{\mathbf{v},t}$ is absolutely continuous with respect to $\overline{\mathbb{P}}^0_{\mathbf{v},t}$ with

$$\frac{d\overline{\mathbb{P}}^\lambda_{\mathbf{v},t}}{d\overline{\mathbb{P}}^0_{\mathbf{v},t}}(\mathbf{y}_{\mathbf{v},[\mathbf{0},\mathbf{t}]}) = \mathbb{E}_{\mathbb{Q}}\left(\overline{\mathrm{M}}^\lambda_t \middle| \mathcal{F}(\mathbf{y}_{\mathbf{v},[\mathbf{0},\mathbf{t}]})\right),$$

where

$$\overline{\mathrm{M}}^\lambda_t(\underline{\omega}) = \exp\left\{\frac{1}{2}\int_0^t \langle Z^\lambda(\mathbf{y}_{\mathbf{v},s}(\underline{\omega})), \mathbf{u}_{\mathbf{v},s}(\underline{\omega})dB_s(\underline{\omega})\rangle_{\mathbf{y}_{\mathbf{v},s}(\underline{\omega})} - \frac{1}{4}\int_0^t \|Z^\lambda(\mathbf{y}_{\mathbf{v},s}(\underline{\omega}))\|^2\, ds\right\}.$$

Consequently we have

$$\begin{aligned}
\lim_{t\to+\infty}(\mathrm{I})^{\mathrm{t}}_\ell &= \lim_{t\to+\infty}\mathbb{E}_{\overline{\mathbb{P}}^0}\left(\frac{1}{\sqrt{t}}\left(d_{\mathcal{W}}(\omega(0),\omega(t)) - t\overline{\ell}_0\right)\frac{d\overline{\mathbb{P}}^\lambda_{\omega(0),t}}{d\overline{\mathbb{P}}^0_{\omega(0),t}}\right)\\
&= \lim_{t\to+\infty}\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{\sqrt{t}}\left(d_{\mathcal{W}}(\mathbf{y}_0,\mathbf{y}_t) - t\overline{\ell}_0\right)\cdot e^{(\mathrm{II})^t_\ell}\right),
\end{aligned}$$

where

$$(\mathrm{II})^t_\ell = \frac{1}{2}\int_0^t \langle Z^\lambda(\mathbf{y}_s(\mathbf{v},\underline{\omega})),\ \mathbf{u}_s(\mathbf{v},\underline{\omega})dB_s\rangle_{\mathbf{y}_s(\mathbf{v},\underline{\omega})} - \frac{1}{4}\int_0^t \|Z^\lambda(\mathbf{y}_s(\mathbf{v},\underline{\omega}))\|^2\, ds.$$

Let $\overline{Z}^\lambda$ be such that $Z^\lambda = \lambda(Z^\lambda)'_0 + \lambda^2\overline{Z}^\lambda$. We calculate for $\lambda = 1/\sqrt{t}$ that

$$\begin{aligned}
(\mathrm{II})^t_\ell &= \frac{1}{2\sqrt{t}}\int_0^t \langle (Z^\lambda)'_0(\mathbf{y}_s),\ \mathbf{u}_s dB_s\rangle_{\mathbf{y}_s} - \frac{1}{4t}\int_0^t \|(Z^\lambda)'_0(\mathbf{y}_s)\|^2\, ds\\
&\quad + \frac{1}{2t}\int_0^t \langle \overline{Z}^\lambda(\mathbf{y}_s),\ \mathbf{u}_s dB_s\rangle_{\mathbf{y}_s}\\
&\quad - \frac{1}{2t^{\frac{3}{2}}}\int_0^t \langle (Z^\lambda)'_0(\mathbf{y}_s), \overline{Z}^\lambda(\mathbf{y}_s)\rangle_{\mathbf{y}_s}\, ds - \frac{1}{4t^2}\int_0^t \|\overline{Z}^\lambda(\mathbf{y}_s)\|^2\, ds\\
&=: \frac{1}{\sqrt{t}}\mathbf{M}^0_t - \frac{1}{2t}\langle \mathbf{M}^0_t, \mathbf{M}^0_t\rangle_t + (\mathrm{III})^{\mathrm{t}}_\ell + (\mathrm{IV})^{\mathrm{t}}_\ell,
\end{aligned}$$

where both $(\mathrm{III})^{\mathrm{t}}_\ell$ and $(\mathrm{IV})^{\mathrm{t}}_\ell$ converge almost surely to zero as t goes to infinity. Therefore, by Proposition 3.1, the variables $\frac{1}{\sqrt{t}}\mathbf{Z}^0_{\ell,t}\cdot\overline{\mathrm{M}}^\lambda_t$ converge in distribution to $\mathbf{Z}^0_\ell e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}$, where $(\mathbf{Z}^0_\ell, \mathbf{M}^0)$ is a bivariate normal variable with covariance matrix Σ_ℓ.

Indeed, to justify

$$\lim_{t\to+\infty}(\mathrm{I})^{\mathrm{t}}_\ell = \lim_{\mathrm{t}\to+\infty}\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{\sqrt{\mathrm{t}}}\mathbf{Z}^0_{\ell,\mathrm{t}}\cdot\overline{\mathrm{M}}^\lambda_{\mathrm{t}}\right) = \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\mathbf{Z}^0_\ell \mathrm{e}^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}\right),$$

we have by Lemma 3.7 that it suffices to show for $q = \frac{3}{2}$,

$$\sup_t \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\left|\frac{1}{\sqrt{t}}\mathbf{Z}^0_{\ell,t}\cdot\overline{\mathrm{M}}^\lambda_t\right|^q\right) < +\infty.$$

By Hölder's inequality, we calculate that

$$\begin{aligned}\left(\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\left|\frac{1}{\sqrt{t}}\mathbf{Z}^0_{\ell,t}\cdot\overline{\mathrm{M}}^\lambda_t\right|^{\frac{3}{2}}\right)\right)^4 \quad &\leq \quad \left(\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{t}\left|\mathbf{Z}^0_{\ell,t}\right|^2\right)\right)^3\cdot\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(e^{6(\mathrm{IV})^t_\ell}\right)\\ &=: \quad (\mathrm{V})^{\mathrm{t}}_\ell\cdot(\mathrm{VI})^{\mathrm{t}}_\ell,\end{aligned}$$

where $(\mathrm{V})^{\mathrm{t}}_\ell$ is uniformly bounded in t by Proposition 3.1. For $(\mathrm{VI})^{\mathrm{t}}_\ell$, we use the Girsanov-Cameron-Martin formula to conclude that

$$\begin{aligned}(\mathrm{VI})^{\mathrm{t}}_\ell \quad &= \quad \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\exp\left(3\int_0^t\langle Z^\lambda(\mathbf{y}_s(\mathbf{v},\underline{\omega})),\ \mathbf{u}_s(\mathbf{v},\underline{\omega})dB_s\rangle_{\mathbf{y}_s(\mathbf{v},\underline{\omega})} - \frac{3}{2}\int_0^t\|Z^\lambda(\mathbf{y}_s(\mathbf{v},\underline{\omega}))\|^2\,ds\right)\right)\\ &\leq \quad \mathbb{E}_{\widetilde{\mathbb{Q}}}\left(\exp\left(\frac{15}{2}\int_0^t\|Z^\lambda(\mathbf{y}_s(\mathbf{v},\underline{\omega}))\|^2\,ds\right)\right)\end{aligned}$$

for some probability measure $\widetilde{\mathbb{Q}}$ on $\overline{\Theta}$. Using again $Z^\lambda = \lambda(Z^\lambda)'_0+\lambda^2\overline{Z}^\lambda$ and that $\lambda = 1/\sqrt{t}$, we see that

$$\int_0^t\|Z^\lambda(\mathbf{y}_s(\mathbf{v},\underline{\omega}))\|^2\,ds \leq \frac{2}{t}\int_0^t\|(Z^\lambda)'_0\|^2\,ds + \frac{2}{t^2}\int_0^t\|\overline{Z}^\lambda\|^2\,ds,$$

where the quantities on the right hand side of the inequality are uniformly bounded in t. So $(\mathrm{VI})^{\mathrm{t}}_\ell$ is uniformly bounded in $t \geq 1$ as well. Now, (3.3) holds. The calculation is the same with $\lambda = -1/\sqrt{t}$.

Finally, since $(\mathbf{Z}^0_\ell, \mathbf{M}^0)$ has a bivariate normal distribution, we have

$$\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\mathbf{Z}^0_\ell e^{\mathbf{M}^0-\frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}\right) = \mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_\ell\mathbf{M}^0),$$

[3] which is $\lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{\ell,t}\mathbf{M}^0_t)$ by Proposition 3.1. □

3.2. **The differential of the stochastic entropy.** For any $\lambda \in (-1,1)$, let $\overline{h}_\lambda$ be the entropy of $\mathcal{L}^\lambda$. In this subsection, we establish the following formula for $(d\overline{h}_\lambda/d\lambda)|_{\lambda=0}$.

Theorem 3.9. *The function $\lambda \mapsto \overline{h}_\lambda$ is differentiable at 0 with*

$$\left.\frac{d\overline{h}_\lambda}{d\lambda}\right|_{\lambda=0} = \lim_{t\to+\infty}\frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{h,t}\mathbf{M}^0_t).$$

[3]We leave the proof of the equality as an exercise. Let the couple (x, y) have a bivariate centered normal distribution. By diagonalizing the covariance matrix, we may assume that

$$\begin{aligned}x \quad &= \quad \cos\theta X - \sin\theta Y\\ y \quad &= \quad \sin\theta X + \cos\theta Y,\end{aligned}$$

where X and Y are independent centered normal distributions with variance σ^2 and τ^2 respectively. Then by independence, all $\mathbb{E}xy, \mathbb{E}y^2$ and $\mathbb{E}xe^y$ are easy to compute and one finds $\mathbb{E}xy = \mathbb{E}(xe^{y-\mathbb{E}y^2/2})$.

Since $c_i, 0 \le i \le 8$, and α_2, α_3 of Lemmas 2.10–2.15 depend only on the geometry of $\widetilde{M}$ and the coefficients of $\mathcal{L}$, we may assume the constants are such that these lemmas hold true for every couple $\mathcal{L}^\lambda, \mathbf{G}^\lambda$ with $\lambda \in (-1, 1)$.

For each $\lambda \in (-1, 1)$, by Lemma 2.22 and the Subadditive Ergodic Theorem we obtain a constant $\overline{h}_{\lambda,0}$ such that for $\overline{\mathbb{P}}^\lambda$-a.e. $\omega \in \Omega_+$,

$$\lim_{t\to+\infty} -\frac{1}{t} \ln G^0_{\mathbf{v}}(\omega(0), \omega(t)) = \overline{h}_{\lambda,0}, \text{ where } \mathbf{v} = \omega(0).$$

For $\overline{\mathbb{P}}^\lambda$-a.e. $\omega \in \Omega_+$, since $\omega(t)$ converges to a point in $\partial\widetilde{M}$ as t tends to infinity, we also have

$$\lim_{t\to+\infty} \frac{1}{t} d_{G^0_{\mathbf{v}}}(\omega(0), \omega(t)) = \overline{h}_{\lambda,0}. \tag{3.6}$$

The equation (3.6) continues to hold if we replace the pathwise limit by its expectation. Actually, since $\overline{\mathbb{P}}^\lambda$ is a probability on Ω_+, using (2.23), we see that

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\sup_{t\in[0,1]} [d_{G^0_{\mathbf{v}}}(\omega(0), \omega(t))]^2\right) \le 2\alpha_2^2 \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\sup_{t\in[0,1]} [d_{\mathcal{W}}(\omega(0), \omega(t))]^2\right) + 8(\ln c_2)^2 < +\infty.$$

So, using (2.22), we have by the Subadditive Ergodic Theorem that for $\lambda \in (-1, 1)$,

$$\lim_{t\to+\infty} \frac{1}{t} \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d_{G^0_{\mathbf{v}}}(\omega(0), \omega(t))\right) = \overline{h}_{\lambda,0}.$$

The main strategy to prove Theorem 3.9 is to split $(\overline{h}_\lambda - \overline{h}_0)/\lambda$ into two terms:

$$\frac{1}{\lambda}(\overline{h}_\lambda - \overline{h}_0) = \frac{1}{\lambda}(\overline{h}_\lambda - \overline{h}_{\lambda,0}) + \frac{1}{\lambda}(\overline{h}_{\lambda,0} - \overline{h}_0) =: (\mathrm{I})^\lambda_h + (\mathrm{II})^\lambda_h,$$

then show $\lim_{\lambda\to0} (\mathrm{I})^\lambda_h = 0$ and $\lim_{\lambda\to0} (\mathrm{II})^\lambda_h = \lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{h,t}\mathbf{M}^0_t)$ successively. Since $d_{G^0_{\mathbf{v}}}$ behaves in the same way as a distance function, the terms $\overline{h}_{\lambda,0}$ and $\overline{h}_0$ are the 'linear drifts' of the diffusions with respect these 'distances' in distributions $\overline{\mathbb{P}}^\lambda$ and $\overline{\mathbb{P}}^0$, respectively. Hence $\lim_{\lambda\to0} (\mathrm{II})^\lambda_h$ can be evaluated by following the evaluation of $(d\overline{\ell}_\lambda/d\lambda)|_{\lambda=0}$ in Section 3.1. The new term $(\mathrm{I})^\lambda_h$ represents the contribution of the of change of Green 'metric' between $G^0_{\mathbf{v}}$ and $G^\lambda_{\mathbf{v}}$. It turns out that this contribution is of order λ^2 for C^1 drift change of $\mathcal{L}^0$. Consequently, we have the following.

Lemma 3.10. *$\lim_{\lambda\to0} (\mathrm{I})^\lambda_h = 0$.*

Proof. For each $\lambda \in (-1, 1)$, recall that by Proposition 2.4 we have for $\overline{\mathbb{P}}^\lambda$-a.e. $\omega \in \Omega_+$, $\omega(0) =: \mathbf{v}$,

$$\begin{aligned} \overline{h}_\lambda &= \lim_{t\to+\infty} -\frac{1}{t} \ln p^\lambda_{\mathbf{v}}(t, \omega(0), \omega(t)) \\ &= \lim_{t\to+\infty} -\frac{1}{t} \int \left(\ln p^\lambda_{\mathbf{v}}(t, x, y)\right) p^\lambda_{\mathbf{v}}(t, x, y)\, dy. \end{aligned} \tag{3.7}$$

Similarly, by the same proof as for Proposition 2.4, we have that

$$\overline{h}_{\lambda,0} = \inf_{s>0}\{\overline{h}_{\lambda,0}(s)\}, \tag{3.8}$$

where for $\overline{\mathbb{P}}^{\lambda}$-a.e. $\omega \in \Omega_+$, $\omega(0) =: \mathbf{v}$,

$$\begin{aligned} \overline{h}_{\lambda,0}(s) &:= \lim_{t\to+\infty} -\frac{1}{t}\ln p^0_{\mathbf{v}}(st, \omega(0), \omega(t)) \\ &= \lim_{t\to+\infty} -\frac{1}{t}\int \left(\ln p^0_{\mathbf{v}}(st, x, y)\right) p^{\lambda}_{\mathbf{v}}(t, x, y)\, dy. \end{aligned} \tag{3.9}$$

Since we are considering the pathwise limit on $p^0_{\mathbf{v}}$ with respect to $\overline{\mathbb{P}}^{\lambda}$, the infimum in (3.8) is not necessarily obtained at $s = 1$. But we still have $\limsup_{\lambda\to 0+}(\mathrm{I})^{\lambda}_h \leq 0$ since

$$\begin{aligned} \overline{h}_{\lambda} - \overline{h}_{\lambda,0} &= \sup_{s>0}\left\{\lim_{t\to+\infty} -\frac{1}{t}\int\left(\ln\frac{p^{\lambda}_{\mathbf{v}}(t,x,y)}{p^0_{\mathbf{v}}(st,x,y)}\right) p^{\lambda}_{\mathbf{v}}(t,x,y)\, dy\right\} \\ &\leq \sup_{s>0}\left\{\lim_{t\to+\infty} \frac{1}{t}\int \frac{p^0_{\mathbf{v}}(st,x,y)}{p^{\lambda}_{\mathbf{v}}(t,x,y)}\, p^{\lambda}_{\mathbf{v}}(t,x,y)\, dy\right\} \\ &= 0, \end{aligned}$$

where we use $-\ln a \leq a^{-1} - 1$ for $a > 0$ to derive the second inequality.

To show $\liminf_{\lambda\to 0+}(\mathrm{I})^{\lambda}_h \geq 0$, we observe that by (3.7), (3.8) and (3.9),

$$\overline{h}_{\lambda} - \overline{h}_{\lambda,0} \geq \overline{h}_{\lambda} - \overline{h}_{\lambda,0}(1) = \lim_{t\to+\infty} -\frac{1}{t}\int\left(\ln\frac{p^{\lambda}_{\mathbf{v}}(t,x,y)}{p^0_{\mathbf{v}}(t,x,y)}\right) p^{\lambda}_{\mathbf{v}}(t,x,y)\, dy. \tag{3.10}$$

We proceed to estimate $\ln(p^{\lambda}_{\mathbf{v}}(t,x,y)/p^0_{\mathbf{v}}(t,x,y))$ using the Girsanov-Cameron-Martin formula in Section 2.6. For $\mathbf{v}, \mathbf{w} \in \widetilde{SM}$, let $\Omega_{\mathbf{v},\mathbf{w},t}$ be the collection of $\omega \in \Omega_+$ such that $\omega(0) = \mathbf{v}, \omega(t) = \mathbf{w}$. Since the space Ω_+ is separable, the measure $\mathbb{P}^{\lambda}$ disintegrates into a class of conditional probabilities $\{\mathbb{P}^{\lambda}_{\mathbf{v},\mathbf{w},t}\}_{\mathbf{v},\mathbf{w}\in\widetilde{SM}}$ on $\Omega_{\mathbf{v},\mathbf{w},t}$'s such that

$$\mathbb{E}_{\mathbb{P}^{\lambda}_{\mathbf{v},\mathbf{w},t}}\left(\frac{d\mathbb{P}^0_t}{d\mathbb{P}^{\lambda}_t}\right) = \frac{\mathbf{p}^0(t,\mathbf{v},\mathbf{w})}{\mathbf{p}^{\lambda}(t,\mathbf{v},\mathbf{w})}. \tag{3.11}$$

Letting $\mathbf{v} = (x,\xi), \mathbf{w} = (y,\xi)$ in (3.11), we obtain

$$\ln\frac{p^0_{\mathbf{v}}(t,x,y)}{p^{\lambda}_{\mathbf{v}}(t,x,y)} = \ln\left(\mathbb{E}_{\mathbb{P}^{\lambda}_{\mathbf{v},\mathbf{w},t}}\left(\frac{d\mathbb{P}^0_{\mathbf{v},t}}{d\mathbb{P}^{\lambda}_{\mathbf{v},t}}\right)\right) \geq \mathbb{E}_{\mathbb{P}^{\lambda}_{\mathbf{v},\mathbf{w},t}}\left(\ln\left(\frac{d\mathbb{P}^0_{\mathbf{v},t}}{d\mathbb{P}^{\lambda}_{\mathbf{v},t}}\right)\right). \tag{3.12}$$

Recall that

$$\frac{d\overline{\mathbb{P}}^0_{\mathbf{v},t}}{d\overline{\mathbb{P}}^{\lambda}_{\mathbf{v},t}}(\mathbf{y}^{\lambda}_{\mathbf{v},[0,t]}) = \mathbb{E}_{\mathbb{Q}^{\lambda}}\left(\overline{\mathrm{M}}^{\lambda}_t \middle| \mathcal{F}(\mathbf{y}^{\lambda}_{\mathbf{v},[0,t]})\right),$$

where $\mathbf{y}^{\lambda} = (\mathbf{y}^{\lambda}_{\mathbf{v},t})_{\mathbf{v}\in SM, t\in\mathbb{R}_+}$ is the diffusion process on $(\overline{\Theta}, \overline{\mathbb{Q}}^{\lambda})$ corresponding to $\mathcal{L}^{\lambda}$ and

$$\overline{\mathrm{M}}^{\lambda}_t(\underline{\omega}) = \exp\left\{-\frac{1}{2}\int_0^t \langle Z^{\lambda}(\mathbf{y}^{\lambda}_{\mathbf{v},s}(\underline{\omega})), \mathbf{u}^{\lambda}_{\mathbf{v},s}(\underline{\omega})dB_s(\underline{\omega})\rangle_{\mathbf{y}^{\lambda}_{\mathbf{v},s}(\underline{\omega})} - \frac{1}{4}\int_0^t \|-Z^{\lambda}(\mathbf{y}^{\lambda}_{\mathbf{v},s}(\underline{\omega}))\|^2\, ds\right\}.$$

So we can further deduce from (3.12) that

$$\begin{aligned}
&\ln \frac{p^0_{\mathbf{v}}(t,x,y)}{p^\lambda_{\mathbf{v}}(t,x,y)}\\
&\geq \mathbb{E}_{\mathbb{P}^\lambda_{\mathbf{v},\mathbf{w},t}}\left(\mathbb{E}_{\mathbb{Q}^\lambda}\left((-\frac{1}{2}\int_0^t \langle Z^\lambda(\mathbf{y}^\lambda_{\mathbf{v},s}), \mathbf{u}^\lambda_{\mathbf{v},s}dB_s\rangle_{\mathbf{y}^\lambda_{\mathbf{v},s}} - \frac{1}{4}\int_0^t \|Z^\lambda(\mathbf{y}^\lambda_{\mathbf{v},s})\|^2\, ds)\big|\mathcal{F}(\mathbf{y}^\lambda_{\mathbf{v},[0,t]})\right)\right)\\
&= -\mathbb{E}_{\mathbb{P}^\lambda_{\mathbf{v},\mathbf{w},t}}\left(\mathbb{E}_{\mathbb{Q}^\lambda}\left((\frac{1}{4}\int_0^t \|Z^\lambda(\mathbf{y}^\lambda_{\mathbf{v},s})\|^2\, ds)\big|\mathcal{F}(\mathbf{y}^\lambda_{\mathbf{v},[0,t]})\right)\right) \geq -\frac{1}{4}(\lambda C)^2 t,
\end{aligned}$$

where the equality holds true since $\int_0^t \langle Z^\lambda(\mathbf{y}^\lambda_{\mathbf{v},s}), \mathbf{u}^\lambda_{\mathbf{v},s}dB_s\rangle_{\mathbf{y}^\lambda_{\mathbf{v},s}}$ is a centered martingale and C is some constant which bounds the norm of $dZ^\lambda/d\lambda$. Reporting this in (3.10) gives

$$\liminf_{\lambda\to 0+} (\mathrm{I})^\lambda_h = \liminf_{\lambda\to 0+} \frac{1}{\lambda}\left(\overline{h}_\lambda - \overline{h}_{\lambda,0}\right) \geq -\frac{1}{4}\limsup_{\lambda\to 0+}(\lambda C^2) = 0.$$

We prove in the same way (with switched arguments) that $\lim_{\lambda\to 0-}(\mathrm{I})^\lambda_h = 0$. □

The analysis of $(\mathrm{II})^\lambda_h$ is analogous to that was used for $(d\overline{\ell}_\lambda/d\lambda)|_{\lambda=0}$. We first find a finite number D_h such that for $\lambda \in [-\delta_1, \delta_1]$ (where δ_1 is from Lemma 3.4) and all $t \in \mathbb{R}_+$,

$$\left|\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(d_{G^0_{\mathbf{v}}}(\omega(0),\omega(t))\right) - t\overline{h}_{\lambda,0}\right| \leq D_h. \tag{3.13}$$

Indeed, using again the fact that the $\mathcal{L}^\lambda$-diffusion has leafwise infinitesimal generator $\mathcal{L}^\lambda_{\mathbf{v}}$ and $\mathbb{P}^\lambda$ is stationary, we have

$$\begin{aligned}
\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(-\ln k^0_{\mathbf{v}}(\omega(t),\xi)\right) &= \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(-\int_0^t \frac{\partial}{\partial s}(\ln k^0_{\mathbf{v}}(\omega(s),\xi))\, ds\right)\\
&= \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(-\int_0^t \mathcal{L}^\lambda_{\mathbf{v}}(\ln k^0_{\mathbf{v}})(\omega(s))\, ds\right)\\
&= -t\int_{M_0\times\partial\widetilde{M}} \mathcal{L}^\lambda_{\mathbf{v}}(\ln k^0_{\mathbf{v}})\, d\widetilde{\mathbf{m}}^\lambda\\
&= t\overline{h}_{\lambda,0}.
\end{aligned}$$

So (3.13) will be a simple consequence of the following lemma.

Lemma 3.11. *There exists a finite number $\widetilde{D}_3$ such that for all $\lambda \in [-\delta_1, \delta_1]$ and $t \in \mathbb{R}_+$,*

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\left|d_{G^0_{\mathbf{v}}}(\omega(0),\omega(t)) + \ln k^0_{\mathbf{v}}(\omega(t),\xi)\right|^2\right) < \widetilde{D}_3.$$

Proof. For $\mathbf{v} = (x,\xi) \in S\widetilde{M}$, $\omega \in \Omega_+$ starting from $\mathbf{v}$, $t \geq 0$, we continue to denote ω its projection to $\widetilde{M}$. Let $z_t(\omega)$ be the point on the geodesic ray $\gamma_{\omega(t),\xi}$ closest to x. We will divide Ω_+ into four events $A'_i(t)$, $1 \leq i \leq 4$, and show there exists a finite $\widetilde{D}'_3$ such that

$$\mathbf{I}_i := \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\left|d_{G^0_{\mathbf{v}}}(\omega(0),\omega(t)) + \ln k^0_{\mathbf{v}}(\omega(t),\xi)\right|^2 \cdot \mathbf{1}_{A'_i(t)}\right) \leq \widetilde{D}'_3.$$

Let $A_1'(t)$ be the event that $d(\omega(0),\omega(t)) > 1$ and $d(\omega(0), z_t(\omega)) \leq 1$. For $\omega \in A_1'(t)$, using Harnack's inequality (2.20) and Lemma 2.14, we easily specify the constant ratios involved in (2.26) and obtain $\mathbf{I}_1 \leq (\ln(c_2c_4^2c_7))^2$.

Let $A_2'(t)$ be the collection of ω such that both $d(\omega(0),\omega(t))$ and $d(\omega(0), z_t(\omega))$ are greater than 1 and $z_t(\omega) \neq \omega(t)$. For such ω, we first have by Lemma 2.15 that

$$\left|d_{G_{\mathbf{v}}^0}(\omega(0),\omega(t)) - d_{G_{\mathbf{v}}^0}(\omega(0), z_t(\omega)) - d_{G_{\mathbf{v}}^0}(\omega(t), z_t(\omega))\right| \leq -\ln c_8. \tag{3.14}$$

For $d_{G_{\mathbf{v}}^0}(\omega(t), z_t(\omega))$, it is true by Lemma 2.14 that

$$\left|d_{G_{\mathbf{v}}^0}(\omega(t), z_t(\omega)) + \ln G_{\mathbf{v}}^0(y,\omega(t)) - \ln G_{\mathbf{v}}^0(y, z_t(\omega))\right| \leq -\ln c_7,$$

where y is an arbitrary point on $\gamma_{z_t(\omega),\xi}$ far away from $z_t(\omega)$. Then we can use Lemma 2.15 to replace $\ln G_{\mathbf{v}}^0(y, z_t(\omega))$ by $\ln G_{\mathbf{v}}^0(y,\omega(0)) - \ln G_{\mathbf{v}}^0(z_t(\omega),\omega(0))$, which, by letting y tend to ξ, gives

$$\left|d_{G_{\mathbf{v}}^0}(\omega(t), z_t(\omega)) + \ln k_{\mathbf{v}}^0(\omega(t),\xi)\right| \leq -\ln(c_7c_8) + \left|\ln G_{\mathbf{v}}^0(\omega(0), z_t(\omega))\right|.$$

This, together with (3.14), further implies

$$\begin{aligned}\left|d_{G_{\mathbf{v}}^0}(\omega(0),\omega(t)) + \ln k_{\mathbf{v}}^0(\omega(t),\xi)\right| &\leq -\ln(c_2c_7c_8^2) + 2\left|\ln G_{\mathbf{v}}^0(\omega(0), z_t(\omega))\right| \\ &\leq -\ln(c_2^5c_7c_8^2) + 2\alpha_2 d(\omega(0), z_t(\omega)).\end{aligned}$$

Since $\widetilde{M}$ is δ-Gromov hyperbolic for some $\delta > 0$, it is true (cf. [**K2**, Proposition 2.1]) that

$$d(x,\gamma_{y,z}) \leq (y|z)_x + 4\delta, \text{ for any } x, y, z \in \widetilde{M}.$$

Consequently, we have

$$d(\omega(0), z_t(\omega)) \leq (\omega(t)|\xi)_{\omega(0)} + 4\delta = \frac{1}{2}\left|d(\omega(0),\omega(t)) - b_{\mathbf{v}}(\omega(t))\right| + 4\delta. \tag{3.15}$$

Using Lemma 3.6, we finally obtain

$$\mathbf{I}_2 \leq 2\left(8\alpha_2\delta - \ln(c_2^5c_7c_8^2)\right)^2 + 2\alpha_2^2 D_3.$$

Let $A_3'(t)$ be the collection of ω such that $d(\omega(0),\omega(t)) > 1$ and $z_t(\omega) = \omega(t)$. Let $\gamma'_{\omega(t),\xi}$ be the two sided extension of the geodesic $\gamma_{\omega(t),\xi}$ and let $z_t'(\omega) \in \gamma'_{\omega(t),\xi}$ be the point closet to $\omega(0)$. Then $z_t'(\omega) \preceq z_t(\omega)$ on $\gamma'_{\omega(t),\xi}$. For $\omega \in A_3'(t)$, using (2.20) if $d(z_t'(\omega),\omega(t)) < 1$ (or using Lemma 2.15, otherwise), we see that

$$\begin{aligned}d_{G_{\mathbf{v}}^0}(\omega(0),\omega(t)) &\leq d_{G_{\mathbf{v}}^0}(\omega(0), z_t'(\omega)) + d_{G_{\mathbf{v}}^0}(z_t'(\omega),\omega(t)) - \ln(c_4c_8) \\ &\leq \alpha_2\left(d(\omega(0), z_t'(\omega)) + d(z_t'(\omega),\omega(t))\right) - \ln(c_2^4c_4c_8) \\ &\leq 3\alpha_2 d(\omega(0),\gamma_{\omega(t),\xi}) - \ln(c_2^4c_4c_8) \\ &\leq \frac{3}{2}\alpha_2\left|d(\omega(0),\omega(t)) - b_{\mathbf{v}}(\omega(t))\right| + 12\alpha_2\delta - \ln(c_2^4c_4c_8),\end{aligned}$$

where we use (3.15) to derive the last inequality. Choose $y \in \gamma_{\omega(t),\xi}$ with $d(\omega(0), y)$ and $d(\omega(t), y)$ are greater than 1. Similarly, using Lemma 2.15, and then Lemma 2.14, we have

$$\begin{aligned}
\left|\ln \frac{G^0_{\mathbf{v}}(\omega(t), y)}{G^0_{\mathbf{v}}(\omega(0), y)}\right| &= \left|d_{G^0_{\mathbf{v}}}(\omega(0), y) - d_{G^0_{\mathbf{v}}}(\omega(t), y)\right| \\
&\leq -\ln c_8 + \left|d_{G^0_{\mathbf{v}}}(\omega(0), z'_t(\omega)) + d_{G^0_{\mathbf{v}}}(z'_t(\omega), y) - d_{G^0_{\mathbf{v}}}(\omega(t), y)\right| \\
&\leq -\ln(c_7 c_8) + d_{G^0_{\mathbf{v}}}(\omega(0), z'_t(\omega)) + d_{G^0_{\mathbf{v}}}(z'_t(\omega), \omega(t)) \\
&\leq \frac{3}{2}\alpha_2 \left|d(\omega(0), \omega(t)) - b_{\mathbf{v}}(\omega(t))\right| + 12\alpha_2\delta - \ln(c_2^4 c_7 c_8).
\end{aligned}$$

Letting y tend to ξ, we obtain

$$\left|\ln k^0_{\mathbf{v}}(\omega(t), \xi)\right| \leq \frac{3}{2}\alpha_2 \left|d(\omega(0), \omega(t)) - b_{\mathbf{v}}(\omega(t))\right| + 12\alpha_2\delta - \ln(c_2^4 c_7 c_8).$$

Thus, using Lemma 3.6 again, we obtain

$$\begin{aligned}
\mathbf{I}_3 &\leq \mathbb{E}_{\overline{\mathbb{P}}^\lambda}\left(\left(3\alpha_2 \left|d(\omega(0), \omega(t)) - b_{\mathbf{v}}(\omega(t))\right| + 24\alpha_2\delta - \ln(c_2^8 c_4 c_7 c_8^2)\right)^2\right) \\
&\leq 18\alpha_2^2 D_3 + 2\left(24\alpha_2\delta - \ln(c_2^8 c_4 c_7 c_8^2)\right)^2.
\end{aligned}$$

Finally, let $A'_4(t)$ be the event that $d(\omega(0), \omega(t)) \leq 1$. Then $\mathbf{I}_4 \leq (-\ln(c_2 c_4))^2$ by the classical Harnack inequality (2.20). $\square$

As before, this reduces the proof of Theorem 3.9 to showing

$$\lim_{t\to+\infty} (\mathrm{III})^t_h = \lim_{t\to+\infty} (1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{h,t}\mathbf{M}^0_t),$$

where

$$(\mathrm{III})^t_h := \mathbb{E}_{\overline{\mathbb{P}}^\lambda_t}\left(\frac{1}{\sqrt{t}}\left(d_{G^0_{\mathbf{v}}}(\omega(0), \omega(t)) - t\overline{h}_0\right)\right). \tag{3.16}$$

The proof is completely parallel to the computation of $\lim_{t\to+\infty} (\mathrm{I})^t_\ell$. We prove Proposition 3.2 first.

Proof of Proposition 3.2. Let $(\mathbf{Z}^1_t)_{t\in\mathbb{R}_+}$, u_1 be as in Proposition 2.18. The process $(\mathbf{Z}^1_t)_{t\in\mathbb{R}_+}$ is a centered martingale with stationary increments and its law under $\overline{\mathbb{P}}^0$ is the same as the law of $(\overline{\mathbf{Z}}^1_t)_{t\in\mathbb{R}_+}$ under $\overline{\mathbb{Q}}^0$, where $(\overline{\mathbf{Z}}^1_t)_{t\in\mathbb{R}_+}$ on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ is given by

$$\overline{\mathbf{Z}}^1_t(\mathbf{v}, \underline{\omega}) = \ln k_{\mathbf{v}}\left(\mathbf{y}_{\mathbf{v},t}(\underline{\omega}), \xi\right) + t\overline{h}_0 + u_1\left(\mathbf{y}_{\mathbf{v},t}(\underline{\omega})\right) - u_1(\mathbf{v}).$$

The pair $(-\overline{\mathbf{Z}}^1_t, \mathbf{M}^0_t)$ is a centered martingale on $(\overline{\Theta}, \overline{\mathbb{Q}}^0)$ with stationary increments and integrable increasing process function. As before, it follows that for $(a, b) \in \mathbb{R}^2$, $-a\overline{\mathbf{Z}}^1_t/\sqrt{t} + b\mathbf{M}^0_t/\sqrt{t}$ converge in distribution in $\overline{\mathbb{Q}}^0$ to a centered normal law with variance $\Sigma_h[a, b] = (a, b)\Sigma_h(a, b)^T$ for some matrix Σ_h. Therefore, $(-\overline{\mathbf{Z}}^1_t/\sqrt{t}, \mathbf{M}^0_t/\sqrt{t})$ converge in distribution to a centered normal vector with covariance Σ_h. Since both $\overline{\mathbf{Z}}_t^1$ and $\mathbf{M}_t^0$ have stationary increments, we also have

$$\Sigma_h[a,b] = \frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}^0}\left[(-a\overline{\mathbf{Z}}_t^1 + b\mathbf{M}_t^0)^2\right], \text{ for all } t \in \mathbb{R}_+.$$

This shows Proposition 3.2 for the pair $(-\overline{\mathbf{Z}}_t^1/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ instead of the pair $(\mathbf{Z}_{\ell,t}^1/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$.

Recall that for $\overline{\mathbb{P}}^0$-a.e. orbits $\omega \in \Omega_+$ with $\omega(0) =: \mathbf{v}$, ω, the projection of ω to $\widetilde{M}$, is such that

$$\limsup_{t\to+\infty} |\ln G_{\mathbf{v}}(x, \omega(t)) - \ln k_{\mathbf{v}}(\omega(t), \xi)| < +\infty.$$

We have by Lemma 3.11 that

$$\sup_t \mathbb{E}_{\overline{\mathbb{P}}^\lambda}(\left|\mathbf{Z}_{h,t}^0 + \overline{\mathbf{Z}}_t^1\right|^2) < +\infty.$$

Therefore,

$$\mathbb{E}_{\overline{\mathbb{P}}^\lambda}(\frac{1}{t}\left|\mathbf{Z}_{h,t}^0 + \overline{\mathbf{Z}}_t^1\right|^2) \to 0, \text{ as } t \to +\infty.$$

Consequently, $(\mathbf{Z}_{h,t}^0/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ has the same limit normal law as $(-\overline{\mathbf{Z}}_t^1/\sqrt{t}, \mathbf{M}_t^0/\sqrt{t})$ and its covariance matrix under $\overline{\mathbb{Q}}^0$ converges to Σ_h as t goes to infinity. □

Finally, Theorem 3.9 follows from

Lemma 3.12. $\lim_{t\to+\infty} (\mathrm{III})_h^t = \lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}_{h,t}^0\mathbf{M}_t^0)$, *where* $(\mathrm{III})_h^t$ *is defined in (3.16).*

Proof. Let $\mathbf{y} = (\mathbf{y}_t)_{t\in\mathbb{R}_+} = (\mathbf{y}_{\mathbf{v},t})_{\mathbf{v}\in SM, t\in\mathbb{R}_+}$ be the diffusion process on $(\overline{\Theta}, \overline{\mathbb{Q}}^\lambda)$ corresponding to $\mathcal{L}^\lambda$ defined in Section 2.6. Using the Girsanov-Cameron-Martin formula for $d\overline{\mathbb{P}}_{\mathbf{v},t}^\lambda/d\overline{\mathbb{P}}_{\mathbf{v},t}^0$ (see (3.1)), we have

$$\begin{aligned}
\lim_{t\to+\infty} (\mathrm{III})_h^t &= \lim_{t\to+\infty} \mathbb{E}_{\overline{\mathbb{P}}^0}\left(\frac{1}{\sqrt{t}}\left(d_{G_{\mathbf{v}}^0}(\omega(0), \omega(t)) - t\overline{h}_0\right)\frac{d\overline{\mathbb{P}}_{\omega(0),t}^\lambda}{d\overline{\mathbb{P}}_{\omega(0),t}^0}\right) \\
&= \lim_{t\to+\infty} \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{\sqrt{t}}\left(d_{G_{\mathbf{v}}^0}(\mathbf{y}_0(\mathbf{v},\underline{\omega}), \mathbf{y}_t(\mathbf{v},\underline{\omega})) - t\overline{h}_0\right)\cdot \overline{\mathrm{M}}_t^\lambda(\underline{\omega})\right) \\
&= \lim_{t\to+\infty} \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{\sqrt{t}}\mathbf{Z}_{h,t}^0 \cdot \overline{\mathrm{M}}_t^\lambda\right),
\end{aligned}$$

where we identify $\mathbf{y}_t(\mathbf{v},\underline{\omega})) \in \widetilde{M} \times \{\xi\}$ with its projection point on $\widetilde{M}$. As before, by Proposition 3.2, the variables $\frac{1}{\sqrt{t}}\mathbf{Z}_{h,t}^0 \cdot \overline{\mathrm{M}}_t^\lambda$ converge in distribution to $\mathbf{Z}_h^0 e^{\mathbf{M}^0 - \frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}$, where $(\mathbf{Z}_h^0, \mathbf{M}^0)$ is a bivariate centered normal variable with covariance matrix Σ_h.

Again, we have by Proposition 3.2 and the same reasoning as in the proof of Lemma 3.8 that

$$\sup_t \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\left|\frac{1}{\sqrt{t}}\mathbf{Z}^0_{h,t}\cdot \overline{\mathrm{M}}^\lambda_t\right|^{\frac{3}{2}}\right) < +\infty.$$

It follows from Lemma 3.7 that

$$\lim_{t\to+\infty}(\mathrm{III})^t_h = \lim_{t\to+\infty}\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\frac{1}{\sqrt{t}}\mathbf{Z}^0_{h,t}\cdot \overline{\mathrm{M}}^\lambda_t\right) = \mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\mathbf{Z}^0_h e^{\mathbf{M}^0-\frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}\right).$$

Finally, using the fact that $(\mathbf{Z}^0_h, \mathbf{M}^0)$ has a bivariate normal distribution, we have again

$$\mathbb{E}_{\overline{\mathbb{Q}}^0}\left(\mathbf{Z}^0_h e^{\mathbf{M}^0-\frac{1}{2}\mathbb{E}_{\overline{\mathbb{Q}}^0}((\mathbf{M}^0)^2)}\right) = \mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_h\mathbf{M}^0),$$

which is $\lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}^0}(\mathbf{Z}^0_{h,t}\mathbf{M}^0_t)$ by Proposition 3.2. □

4. Infinitesimal Morse correspondence

In this section, we study the limit (1.3) and give an expression for the derivative of the geodesic spray when the metric varies in $\Re(M)$.

Let (M, g) be a negatively curved closed connected m-dimensional Riemannian manifold as before. Let $\partial\widetilde{M}$ be the geometric boundary of the universal cover space $(\widetilde{M}, \widetilde{g})$. We can identify $\widetilde{M}\times\partial\widetilde{M}$ with $S\widetilde{M}_{\widetilde{g}}$, the unit tangent bundle of $\widetilde{M}$ in metric $\widetilde{g}$, by sending (x, ξ) to the unit tangent vector of the $\widetilde{g}$-geodesic starting at x pointing at ξ.

Let $\lambda \in (-1, 1) \mapsto g^\lambda$ be a one-parameter family of C^3 metrics on M of negative curvature with $g^0 = g$. Denote by $\widetilde{g}^\lambda$ the G-invariant extension of g^λ to $\widetilde{M}$. For each λ, the geometric boundary of $(\widetilde{M}, \widetilde{g}^\lambda)$, denoted $\partial\widetilde{M}_{\widetilde{g}^\lambda}$, can be identified with $\partial\widetilde{M}$ since the identity isomorphism from $G = \pi_1(M)$ to itself induces a homeomorphism between $\partial\widetilde{M}_{\widetilde{g}^\lambda}$ and $\partial\widetilde{M}$. So each $(x, \xi) \in \widetilde{M}\times\partial\widetilde{M}$ is also associated with the $\widetilde{g}^\lambda$-geodesic spray $\overline{X}_{\widetilde{g}^\lambda}(x, \xi)$, the horizontal vector in $TT\widetilde{M}$ which projects to the unit tangent vector of the $\widetilde{g}^\lambda$-geodesic starting at x pointing towards ξ. Our very first step to study the differentiability of the linear drift under a one-parameter family of conformal changes g^λ of g is to understand the differentiable dependence of the geodesic sprays $\overline{X}_{\widetilde{g}^\lambda}(x, \xi)$ on the parameter λ.

For each g^λ, there exist (g, g^λ)-*Morse correspondence* ([**Ano1**, **Gro**, **Mor**]), the homeomorphisms from SM_g to SM_{g^λ} sending a g geodesic on M to a g^λ geodesic on M. The (g, g^λ)-Morse correspondence is not unique, but any two such maps only differ by shifts in the geodesic flow directions (i.e., if F_1, F_2 are two (g, g^λ)-Morse correspondence maps, then there exists a real valued function $t(\cdot)$ on SM_g such that $F_1^{-1}\circ F_2(v) = \Phi_{t(v)}(v)$ for $v \in SM_g$), where Φ is the geodesic flow map on SM_g ([**Ano1**, **Gro**, **Mor**], see [**FF**, Theorem 1.1]).

Let us construct a (g, g^λ)-Morse correspondence map by lifting the systems to their universal cover spaces as in [**Gro**]. For an oriented geodesic γ in $(\widetilde{M}, \widetilde{g})$, denote by $\partial^+(\gamma) \in \partial\widetilde{M}_{\widetilde{g}}$ and $\partial^-(\gamma) \in \partial\widetilde{M}_{\widetilde{g}}$ the asymptotic classes of its positive and negative directions. The map $\gamma \mapsto (\partial^+(\gamma), \partial^-(\gamma)) \in \partial\widetilde{M}_{\widetilde{g}} \times \partial\widetilde{M}_{\widetilde{g}}$ establishes a homeomorphism between the set of all oriented geodesics in $(\widetilde{M}, \widetilde{g})$ and $\partial^2(\widetilde{M}_{\widetilde{g}}) = (\partial\widetilde{M}_{\widetilde{g}} \times \partial\widetilde{M}_{\widetilde{g}})\backslash\{(\xi, \xi) : \ \xi \in \partial\widetilde{M}_{\widetilde{g}}\}$. So the natural homeomorphism $\mathrm{D}^\lambda : \ \partial^2(\widetilde{M}_{\widetilde{g}}) \to \partial^2(\widetilde{M}_{\widetilde{g}^\lambda})$ induced from the identity isomorphism from G to itself can be viewed as a homeomorphism between the sets of oriented geodesics in $(\widetilde{M}, \widetilde{g})$ and $(\widetilde{M}, \widetilde{g}^\lambda)$. Realize points from $S\widetilde{M}_{\widetilde{g}}$ by pairs (γ, y), where γ is an oriented geodesic and $y \in \gamma$, and define a map $\widetilde{F}^\lambda : \ S\widetilde{M}_{\widetilde{g}} \to S\widetilde{M}_{\widetilde{g}^\lambda}$ by sending $(\gamma, y) \in S\widetilde{M}_{\widetilde{g}}$ to

$$\widetilde{F}^\lambda(\gamma, y) = (\mathrm{D}^\lambda(\gamma), y'),$$

where y' is the intersection point of $\mathrm{D}^\lambda(\gamma)$ and the hypersurface $\{\exp_{\widetilde{g}} Y : \ Y \perp \mathrm{v}\}$, where v is the vector in $S_y\widetilde{M}_{\widetilde{g}}$ pointing at $\partial^+(\gamma)$. The map $\widetilde{F}^\lambda$ is a homeomorphism since both g and g^λ are of negative curvature. Returning to SM_g and SM_{g^λ}, we obtain a map F^λ. Given any sufficiently small ϵ, if g^λ is in a sufficiently small C^3-neighborhood of g, then F^λ is the only (g, g^λ)-Morse correspondence map such that the footpoint of $F^\lambda(v)$ belongs to the hypersurface of points $\{\exp_g Y : \ Y \perp v, \|Y\|_g < \epsilon\}$.

Regard SM_{g^λ} as a subset of TM and let $\pi^\lambda : \ SM_{g^\lambda} \to SM_g$ be the projection map sending v to $v/\|v\|_g$. The map π^λ records the direction information of the vectors of SM_{g^λ} in SM_g. Let $F^\lambda : SM_g \to SM_{g^\lambda}$ be the (g, g^λ)-Morse correspondence map obtained as above. We obtain a one-parameter family of homeomorphisms $\pi^\lambda \circ F^\lambda$ from SM_g to SM_g. By using the implicit function theorem, de la Llave-Marco-Moriyón [**LMM**, Theorem A.1] improved the differentiable dependence of $\pi^\lambda \circ F^\lambda$ on the parameter λ.

Theorem 4.1. *(cf.* [**FF**, Theorem 2.1]*) There exists a C^3 neighborhood of g so that for any C^3 one-parameter family of C^3 metrics $\lambda \in (-1, 1) \mapsto g^\lambda$ in it with $g^0 = g$, the map $\lambda \mapsto \pi^\lambda \circ F^\lambda$ is C^3 with values in the Banach manifold of continuous maps $SM_g \to SM_g$. The tangent to the curve $\pi^\lambda \circ F^\lambda$ is a continuous vector field Ξ_λ on SM_g.*

Following Fathi-Flaminio [**FF**], we will call $\Xi := \Xi_0$ in Theorem 4.1 the *infinitesimal Morse correspondence at g for the curve g^λ*. It was shown in [**FF**] that the vector field Ξ only depends on g and the differential of g^λ in λ at 0. More precisely, the horizontal and the vertical components of Ξ are described by:

Theorem 4.2. *(*[**FF**, Proposition 2.7]*) Let Ξ be the infinitesimal Morse correspondence at g for the curve g^λ and let Ξ_γ be the restriction of horizontal component of Ξ to a unit speed g-geodesic γ. Then Ξ_γ is the unique bounded solution of the equation*

$$\nabla^2_{\dot\gamma}\Xi_\gamma + \mathrm{R}(\Xi_\gamma, \dot\gamma)\dot\gamma + \Gamma_{\dot\gamma}\dot\gamma - \langle \Gamma_{\dot\gamma}\dot\gamma, \dot\gamma\rangle\dot\gamma = 0 \tag{4.1}$$

satisfying $\langle \Xi_\gamma, \dot\gamma\rangle = 0$ *along* γ*, where* $\dot\gamma(t) = \frac{d}{dt}\gamma(t)$*,* ∇ *and* R *are the Levi-Civita connection and curvature tensor of metric* g*,* ∇^λ *is the Levi-Civita connection of the metric* g^λ *and* $\Gamma = \partial_\lambda \nabla^\lambda|_{\lambda=0}$*. The vertical component of* Ξ *in* $T(SM_g)$ *is given by* $\nabla_{\dot\gamma}\Xi_\gamma$*.*

We will still denote by Ξ the G-invariant extension to $T(S\widetilde{M}_{\widetilde{g}})$ of the infinitesimal Morse-correspondence at g for the curve g^λ. For any geodesic γ in $(\widetilde{M}, \widetilde{g})$, let $N(\gamma)$ be the normal bundle of γ:

$$N(\gamma) = \cup_{t\in\mathbb{R}} N_t(\gamma), \text{ where } N_t(\gamma) \ = \ (\dot\gamma(t))^\perp \ = \ \{E \in T_{\gamma(t)}\widetilde{M} : \langle E, \dot\gamma(t)\rangle = 0\}.$$

The one-parameter family of vectors along γ arising in equation (4.1)

$$\Upsilon(t) := \left(\Gamma_{\dot\gamma}\dot\gamma - \langle \Gamma_{\dot\gamma}\dot\gamma, \dot\gamma\rangle\dot\gamma\right)(\gamma(t)), \ t\in\mathbb{R}, \tag{4.2}$$

is such that $\Upsilon(t)$ belongs to $N_t(\gamma)$ for all t. The restriction of the infinitesimal Morse correspondence to γ is $(\Xi_\gamma, \nabla_{\dot\gamma}\Xi_\gamma)$, with both Ξ_γ and $\nabla_{\dot\gamma}\Xi_\gamma$ belonging to $N(\gamma)$ as well. In the following, we will specify Ξ_γ and $\nabla_{\dot\gamma}\Xi_\gamma$ using Υ and a special coordinate system of $N_t(\gamma)$'s arising from the stable and unstable Jacobi fields along γ.

Let $\mathbf{v} = (x, \mathrm{v})$ be a point in $T\widetilde{M}$. Recall from Subsection 2.3 the definition (2.14) of Jacobi Fields, Jacobi tensors and, for $\mathbf{v} \in S\widetilde{M}$, of the stable and unstable tensors along $\gamma_{\mathbf{v}}$ denoted $S_{\mathbf{v}}$ and $U_{\mathbf{v}}$.

For each $\mathbf{v} \in S\widetilde{M}$, the vectors $(Y, S'_{\mathbf{v}}(0)Y)$, $Y \in N_0(\gamma)$, (or $(Y, U'_{\mathbf{v}}(0)Y)$) generate $TW^{ss}_{\mathbf{v}}$ (or $TW^{su}_{\mathbf{v}}$). As a consequence of the Anosov property of the geodesic flow on $S\widetilde{M}$, the operator $(U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))$ is positive and symmetric (see [**Bo**]). Hence we can choose vectors $\vec{x}_1, \cdots, \vec{x}_{m-1}$ to form a basis of $N_0(\gamma_{\mathbf{v}})$ so that

$$\langle (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))\vec{x}_i, \vec{x}_j\rangle = \delta_{ij}. \tag{4.3}$$

Let $J_1, \cdots, J_{2m-2}$ be the Jacobi fields with

$$(J_i(0), J'_i(0)) = \begin{cases} (\vec{x}_i, S'_{\mathbf{v}}(0)\vec{x}_i), & \text{if } i \in \{1, \cdots, m-1\}; \\ (\vec{x}_{i+1-m}, U'_{\mathbf{v}}(0)\vec{x}_{i+1-m}), & \text{if } i \in \{m, \cdots, 2m-2\}. \end{cases}$$

Since the Wronskian of two Jacobi fields remains constant along geodesics, we have

$$W(J_i, J_j) = \begin{cases} 0, & \text{if } i,j \in \{1, \cdots, m-1\} \text{ or } i,j \in \{m, \cdots, 2m-2\}; \\ -\delta_{i,j+1-m}, & \text{if } i \in \{1, \cdots, m-1\} \ \text{ and } j \in \{m, \cdots, 2m-2\}. \end{cases} \tag{4.4}$$

Equivalently, if we write $\mathbf{J}_{\mathrm{s}}$ for the matrix with column vectors $(J_1, \cdots, J_{m-1})$ and $\mathbf{J}_{\mathrm{u}}$ for the matrix with column vectors $(J_m, \cdots, J_{2m-2})$, then (4.4) gives

$$\mathbf{J}^*_w \mathbf{J}'_w = (\mathbf{J}'_w)^* \mathbf{J}_w, \ w = \mathrm{s} \text{ or } \mathrm{u}, \text{ and } \ \mathbf{J}^*_{\mathrm{u}}\mathbf{J}'_{\mathrm{s}} - (\mathbf{J}'_{\mathrm{u}})^*\mathbf{J}_{\mathrm{s}} = -\mathrm{Id}. \tag{4.5}$$

The collection $(J_i(t), J'_i(t)), i = 1, \cdots, m-1$, (or $(J_i(t), J'_i(t)), i = m, \cdots, 2m-2$) generate $TW^{ss}_{\dot\gamma_{\mathbf{v}}(t)}$ (or $TW^{su}_{\dot\gamma_{\mathbf{v}}(t)}$). Consequently, any $V(t) = (V_1(t), V_2(t)) \in TT\widetilde{M}$ along γ with

$V_i(t) \in N_t(\gamma)$, $i = 1, 2$, can be expressed as

$$V_1(t) = \sum_{i=1}^{m-1} a_i(t)(J_i(t), J_i'(t)), \quad V_2(t) = \sum_{i=m}^{2m-2} b_{i-m+1}(t)(J_i(t), J_i'(t)),$$

where $(a_i(t), b_i(t)), i = 1 \cdots m-1$, are $2m-2$ real numbers. Writing them as two column vectors $\vec{a}(t), \vec{b}(t)$, we write any such $V(t)$ as

$$V(t) = (\mathbf{J}_{\mathrm{s}}(t)\vec{a}(t), \mathbf{J}_{\mathrm{s}}'(t)\vec{a}(t)) + (\mathbf{J}_{\mathrm{u}}(t)\vec{b}(t), \mathbf{J}_{\mathrm{u}}'(t)\vec{b}(t)).$$

To specify the infinitesimal Morse correspondence Ξ at g for the curve g^λ, it suffices to find the coefficients $\vec{a}(t), \vec{b}(t)$ for the restriction of Ξ along any $\widetilde{g}$-geodesic γ.

Proposition 4.3. *Let Ξ be the infinitesimal Morse correspondence at g for a C^3 one-parameter family of C^3 metrics g^λ with $g^0 = g$. Then the restriction of Ξ to a $\widetilde{g}$-geodesic γ is $(\mathbf{J}_{\mathrm{s}}(t)\vec{a}(t), \mathbf{J}_{\mathrm{s}}'(t)\vec{a}(t)) + (\mathbf{J}_{\mathrm{u}}(t)\vec{b}(t), \mathbf{J}_{\mathrm{u}}'(t)\vec{b}(t))$ with*

$$\vec{a}(t) = \int_{-\infty}^{t} \mathbf{J}_{\mathrm{u}}^*(s)\Upsilon(s)\, ds, \quad \vec{b}(t) = \int_{t}^{+\infty} \mathbf{J}_{\mathrm{s}}^*(s)\Upsilon(s)\, ds, \tag{4.6}$$

where $\Upsilon(s)$ is given by (4.2).

Proof. By the construction of Morse correspondence, for any $\widetilde{g}$-geodesic γ, the value of Ξ along γ, denoted $\Xi(\gamma)$, belongs to $N(\gamma) \times N(\gamma)$. So, there are $\vec{a}(t), \vec{b}(t), t \in \mathbb{R}$, such that

$$\Xi(\gamma) = (\mathbf{J}_{\mathrm{s}}(t)\vec{a}(t), \mathbf{J}_{\mathrm{s}}'(t)\vec{a}(t)) + (\mathbf{J}_{\mathrm{u}}(t)\vec{b}(t), \mathbf{J}_{\mathrm{u}}'(t)\vec{b}(t)).$$

The horizontal part Ξ_γ of $\Xi(\gamma)$ is $\mathbf{J}_{\mathrm{s}}(t)\vec{a}(t) + \mathbf{J}_{\mathrm{u}}(t)\vec{b}(t)$. On the other hand, the vertical part of $\Xi(\gamma)$ is $\mathbf{J}_{\mathrm{s}}'(t)\vec{a}(t) + \mathbf{J}_{\mathrm{u}}'(t)\vec{b}(t)$, which, by Theorem 4.2, is also

$$\nabla_{\dot\gamma}\Xi_\gamma = \mathbf{J}_{\mathrm{s}}'(t)\vec{a}(t) + \mathbf{J}_{\mathrm{u}}'(t)\vec{b}(t) + \mathbf{J}_{\mathrm{s}}(t)\vec{a}'(t) + \mathbf{J}_{\mathrm{u}}(t)\vec{b}'(t).$$

So we must have

$$\mathbf{J}_{\mathrm{s}}(t)\vec{a}'(t) + \mathbf{J}_{\mathrm{u}}(t)\vec{b}'(t) = 0. \tag{4.7}$$

Differentiating $\nabla_{\dot\gamma}\Xi_\gamma = \mathbf{J}_{\mathrm{s}}'(t)\vec{a}(t) + \mathbf{J}_{\mathrm{u}}'(t)\vec{b}(t)$ along γ and reporting it in (4.1), we obtain

$$\mathbf{J}_{\mathrm{s}}'(t)\vec{a}'(t) + \mathbf{J}_{\mathrm{u}}'(t)\vec{b}'(t) + \mathbf{J}_{\mathrm{s}}''(t)\vec{a}(t) + \mathbf{J}_{\mathrm{u}}''(t)\vec{b}(t) + \mathrm{R}(t)\mathbf{J}_{\mathrm{s}}(t)\vec{a}(t) + \mathrm{R}(t)\mathbf{J}_{\mathrm{u}}(t)\vec{b}(t) = -\Upsilon(t),$$

which simplifies to

$$\mathbf{J}_{\mathrm{s}}'(t)\vec{a}'(t) + \mathbf{J}_{\mathrm{u}}'(t)\vec{b}'(t) = -\Upsilon(t) \tag{4.8}$$

by the defining property of Jacobi fields. Using (4.5), we solve $\vec{a}', \vec{b}'$ from (4.7), (4.8) with

$$\vec{a}' = \mathbf{J}_{\mathrm{u}}^*\Upsilon, \ \vec{b}' = -\mathbf{J}_{\mathrm{s}}^*\Upsilon. \tag{4.9}$$

Note that $\mathbf{J}_{\mathrm{u}}(-\infty) = \mathbf{J}_{\mathrm{s}}(+\infty) = 0$. Finally, we recover $\vec{a}(t), \vec{b}(t)$ from (4.9) by integration. □

For any $s \in \mathbb{R}$, let $(\mathrm{K}_s, \mathrm{K}'_s)$ be the unique Jacobi field along a $\widetilde{g}$-geodesic γ such that

$$\mathrm{K}'_s(s) = \Upsilon(s) \ \text{ and } \ \mathrm{K}_s(s) = 0.$$

Then

$$\left(\mathrm{K}_s(0), \mathrm{K}'_s(0)\right) = (D\mathbf{\Phi}_s)^{-1}\left(0, \Upsilon(s)\right).$$

We further express Ξ using K_s's by specifying the value of $\Xi(\gamma(0))$ for any $\widetilde{g}$-geodesic γ.

Proposition 4.4. *Let Ξ be the infinitesimal Morse correspondence at g for a C^3 one-parameter family of C^3 metrics g^λ with $g^0 = g$. Then for the $\widetilde{g}$-geodesic γ with $\dot{\gamma}(0) = \mathbf{v}$:*

$$\begin{aligned}
\Xi_\gamma(0) &= (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1}\Big[\int_{-\infty}^{0} (\mathrm{K}'_s(0) - U'_{\mathbf{v}}(0)\mathrm{K}_s(0))\ ds \\
&\qquad\qquad + \int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0))\ ds\Big], \\
(\nabla_{\dot{\gamma}}\Xi_\gamma)(0) &= S'_{\mathbf{v}}(0)(U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1}\int_{-\infty}^{0} (\mathrm{K}'_s(0) - U'_{\mathbf{v}}(0)\mathrm{K}_s(0))\ ds \\
&\quad + U'_{\mathbf{v}}(0)(U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))^{-1}\int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0))\ ds.
\end{aligned}$$

Proof. By Proposition 4.3, for any $\widetilde{g}$-geodesic γ,

$$\Xi(\gamma(0)) = (\Xi_\gamma(0), (\nabla_{\dot{\gamma}}\Xi_\gamma)(0)) = \left(\mathbf{J}_{\mathrm{s}}(0)\vec{a}(0) + \mathbf{J}_{\mathrm{u}}(0)\vec{b}(0), \mathbf{J}'_{\mathrm{s}}(0)\vec{a}(0) + \mathbf{J}'_{\mathrm{u}}(0)\vec{b}(0)\right),$$

where $\vec{a}(0), \vec{b}(0)$ are given by (4.6). We first express $\vec{a}(0)$ using K_s's. Let $s \leq 0$. The Wronskian between K_s and any unstable Jacobi fields are preserved along the geodesics and must have the same value at $\gamma(s)$ and $\gamma(0)$. This gives

$$\mathbf{J}^*_{\mathrm{u}}(s)\Upsilon(s) = \mathbf{J}^*_{\mathrm{u}}(0)\mathrm{K}'_s(0) - (\mathbf{J}'_{\mathrm{u}})^*(0)\mathrm{K}_s(0).$$

Consequently,

$$(\mathbf{J}^*_{\mathrm{u}})^{-1}(0)\mathbf{J}^*_{\mathrm{u}}(s)\Upsilon(s) = \mathrm{K}'_s(0) - (\mathbf{J}^*_{\mathrm{u}})^{-1}(0)(\mathbf{J}'_{\mathrm{u}})^*(0)\mathrm{K}_s(0) = \mathrm{K}'_s(0) - U'_{\mathbf{v}}(0)\mathrm{K}_s(0),$$

where we use the fact that $\mathbf{J}'_{\mathrm{u}}(0) = U'_{\mathbf{v}}(0)\mathbf{J}_{\mathrm{u}}(0)$ for the second equality. So we have

$$\vec{a}(0) = \mathbf{J}^*_{\mathrm{u}}(0)\int_{-\infty}^{0} (\mathrm{K}'_s(0) - U'_{\mathbf{v}}(0)\mathrm{K}_s(0))\ ds.$$

Similarly, for any $s \geq 0$, a comparison of the Wronskian between K_s and any stable Jacobi fields at time s and 0 gives

$$\mathbf{J}^*_{\mathrm{s}}(s)\Upsilon(s) = \mathbf{J}^*_{\mathrm{s}}(0)\mathrm{K}'_s(0) - (\mathbf{J}'_{\mathrm{s}})^*(0)\mathrm{K}_s(0).$$

As a consequence, we have

$$(\mathbf{J}^*_{\mathrm{s}})^{-1}(0)\mathbf{J}^*_{\mathrm{s}}(s)\Upsilon(s) = \mathrm{K}'_s(0) - (\mathbf{J}^*_{\mathrm{s}})^{-1}(0)(\mathbf{J}'_{\mathrm{s}})^*(0)\mathrm{K}_s(0) = \mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0),$$

which gives

$$\vec{b}(0) = \mathbf{J}^*_{\mathrm{s}}(0)\int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0))\ ds.$$

The formula for $\Xi(\gamma(0))$ follows by using $\mathbf{J}_{\mathrm{s}}(0)=\mathbf{J}_{\mathrm{u}}(0)$ and $\mathbf{J}_{\mathrm{u}}(0)\mathbf{J}_{\mathrm{u}}^{*}(0)=(U'_{\mathbf{v}}(0)-S'_{\mathbf{v}}(0))^{-1}$. $\square$

A dynamical point of view of the integrability of the integrals in Proposition 4.4 is that $(\mathrm{K}'_s(0)-U'_{\mathbf{v}}(0)\mathrm{K}_s(0))$ $(s\le 0)$ is the stable vertical part of $(D\mathbf{\Phi}_s)^{-1}(0,\Upsilon(s))$ and hence decays exponentially when s goes to $-\infty$, while $(\mathrm{K}'_s(0)-S'_{\mathbf{v}}(0)\mathrm{K}_s(0))$ $(s\ge 0)$ is the unstable vertical part of $(D\mathbf{\Phi}_s)^{-1}(0,\Upsilon(s))$ and thus decays exponentially when s goes to $+\infty$.

For any curve $\lambda\in(-1,1)\mapsto\mathcal{C}_\lambda\in\mathbf{N}$ (or $\mathcal{C}^\lambda\in\mathbf{N}$) on some Riemannian manifold $\mathbf{N}$, we write $(\mathcal{C}_\lambda)'_0:=(d\mathcal{C}_\lambda/d\lambda)|_{\lambda=0}$ (or $(\mathcal{C}^\lambda)'_0:=(d\mathcal{C}^\lambda/d\lambda)|_{\lambda=0}$) whenever the differential exists. We can put a formula concerning $\left(\overline{X}_{\widetilde{g}^\lambda}\right)'_0$ for any C^3 curve g^λ in $\Re(M)$ with $g^0=g$.

Proposition 4.5. *Let (M,g) be a negatively curved closed connected m-dimensional Riemannian manifold. Then for any C^3 one-parameter family of C^3 metrics $\lambda\in(-1,1)\mapsto g^\lambda$ in it with $g^0=g$, the map $\lambda\mapsto\overline{X}_{\widetilde{g}^\lambda}(x,\xi)$ is differentiable at $\lambda=0$ for each $\mathbf{v}=(x,\xi)$ with*

$$\left(\overline{X}_{\widetilde{g}^\lambda}\right)'_0(x,\xi)=\left(0,\left(\|\overline{X}_{\widetilde{g}^\lambda}\|_{\widetilde{g}}\right)'_0(\mathbf{v})\mathbf{v}+\int_0^{+\infty}(\mathrm{K}'_s(0)-S'_{\mathbf{v}}(0)\mathrm{K}_s(0))\,ds\right).$$

Proof. Express the homeomorphism $\widetilde{F}^\lambda$ as a map from $\widetilde{M}\times\partial\widetilde{M}$ to $\widetilde{M}\times\partial\widetilde{M}_{\widetilde{g}^\lambda}$ with

$$\widetilde{F}^\lambda(x,\xi)=(f^\lambda_\xi(x),\xi),\ \forall(x,\xi)\in S\widetilde{M},$$

where f^λ_ξ records the change of footpoint of the (g,g^λ)-Morse correspondence $\widetilde{F}^\lambda$. We have

$$\begin{aligned}&\frac{1}{\lambda}\left(\overline{X}_{\widetilde{g}^\lambda}(x,\xi)-\overline{X}_{\widetilde{g}}(x,\xi)\right)\\&=\frac{1}{\lambda}\left(\overline{X}_{\widetilde{g}^\lambda}(x,\xi)-\frac{\overline{X}_{\widetilde{g}^\lambda}(x,\xi)}{\|\overline{X}_{\widetilde{g}^\lambda}(x,\xi)\|_{\widetilde{g}}}\right)+\frac{1}{\lambda}\left(\frac{\overline{X}_{\widetilde{g}^\lambda}(x,\xi)}{\|\overline{X}_{\widetilde{g}^\lambda}(x,\xi)\|_{\widetilde{g}}}-\overline{X}_{\widetilde{g}}(x,\xi)\right)\\&=:(a)_\lambda+(b)_\lambda.\end{aligned}$$

When λ tends to zero, $(a)_\lambda$ tends to $(0,\left(\|\overline{X}_{\widetilde{g}^\lambda}\|\right)'_0(\mathbf{v})\mathbf{v})$. For $(b)_\lambda$, we can transport $\overline{X}_{\widetilde{g}}(x,\xi)$ to $\dfrac{\overline{X}_{\widetilde{g}^\lambda}(x,\xi)}{\|\overline{X}_{\widetilde{g}^\lambda}(x,\xi)\|_{\widetilde{g}}}$ along two pieces of curves: the first is to follow the footpoint of the inverse of the (g^λ,g)-Morse correspondence from $\overline{X}_{\widetilde{g}}(x,\xi)$ to $\overline{X}_{\widetilde{g}}((f^\lambda_\xi)^{-1}(x),\xi)$ with the constraint that the vector remains within $TW^s(x,\xi)$; the second is to use the (g^λ,g)-Morse correspondence from $\overline{X}_{\widetilde{g}}((f^\lambda_\xi)^{-1}(x),\xi)$ to $\dfrac{\overline{X}_{\widetilde{g}^\lambda}(x,\xi)}{\|\overline{X}_{\widetilde{g}^\lambda}(x,\xi)\|_{\widetilde{g}}}$. By Theorem 4.1 and Theorem 4.2, the second curve is C^1 and the derivative is $(\Xi_{\gamma_{\mathbf{v}}}(0),\nabla_{\dot\gamma_{\mathbf{v}}}\Xi_{\gamma_{\mathbf{v}}}(0))$, which is also $(\mathbf{J}_{\mathrm{s}}(0)\vec{a}(0),\mathbf{J}'_{\mathrm{s}}(0)\vec{a}(0))+(\mathbf{J}_{\mathrm{u}}(0)\vec{b}(0),\mathbf{J}'_{\mathrm{u}}(0)\vec{b}(0))$ with $\vec{a}(0),\vec{b}(0)$ from Proposition 4.3. The horizontal projection of the first curve is the reverse of the second one; so it is also C^1 and the horizontal part of the derivative is $-\Xi_{\gamma_{\mathbf{v}}}(0)$. Since it belongs to $TW^s(x,\xi)$ which is a graph over the horizontal plane, the vertical part is also C^1 and the derivative is given by $S'_{\mathbf{v}}(0)(-\Xi_{\gamma_{\mathbf{v}}}(0))$. So,

$$\lim_{\lambda\to 0}(b)_\lambda = \big(0, (\nabla_{\dot\gamma_{\mathbf{v}}}\Xi_{\gamma_{\mathbf{v}}})(0) - S'_{\mathbf{v}}(0)\Xi_{\gamma_{\mathbf{v}}}(0)\big) = \left(0, (U'_{\mathbf{v}}(0) - S'_{\mathbf{v}}(0))\mathbf{J}_{\mathrm{u}}(0)\vec{b}(0)\right),$$

which, by our choice of $\mathbf{J}_{\mathrm{u}}(0) = \mathbf{J}_{\mathrm{s}}(0)$ and the defining property of $\mathbf{J}_{\mathrm{u}}(0)$ in (4.3), is

$$\left(0, (\mathbf{J}_{\mathrm{s}}^*)^{-1}(0)\vec{b}(0)\right) = \left(0, \int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0))\, ds\right).$$

□

Corollary 4.6. *Let (M, g) be a negatively curved closed connected Riemannian manifold and let $\lambda \in (-1, 1) \mapsto g^\lambda \in \Re(M)$ be a C^3 curve of C^3 conformal changes of the metric $g^0 = g$. The map $\lambda \mapsto \overline{X}_{\widetilde{g}^\lambda}(x, \xi)$ is differentiable for each $\mathbf{v} = (x, \xi)$ with*

$$\left(\overline{X}_{\widetilde{g}^\lambda}\right)'_0(x, \xi) = \left(0, -\varphi\circ\varpi\ \mathbf{v} + \int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0))\, ds\right),$$

where $\varphi:\ M \to \mathbb{R}$ is such that $g^\lambda = e^{2\lambda\varphi + O(\lambda^2)}g$, ϖ denotes the projections $\varpi : SM \to M$ and $\varpi : S\widetilde{M} \to \widetilde{M}$, and $(\mathrm{K}_s(0), \mathrm{K}'_s(0)) = (D\mathbf{\Phi}_s)^{-1}(0, \Upsilon(s))$ with $\Upsilon = -\nabla\varphi + \langle\nabla\varphi, \dot\gamma_{\mathbf{v}}\rangle\dot\gamma_{\mathbf{v}}$.

Proof. Let $\lambda \in (-1, 1) \mapsto \varphi^\lambda$ be such that $g^\lambda = e^{2\varphi^\lambda}g$. Clearly, $\|\overline{X}_{\widetilde{g}^\lambda}\|_{\widetilde{g}} = e^{-\varphi^\lambda\circ\varpi}$ and hence $\left(\|\overline{X}_{\widetilde{g}^\lambda}\|_{\widetilde{g}}\right)'_0(\mathbf{v})\mathbf{v} = -\varphi\circ\varpi\ \mathbf{v}$. Write $\langle\cdot,\cdot\rangle_\lambda$ for the $\widetilde{g}^\lambda$-inner product and let ∇^λ denote the associated Levi-Civita connection (we simply write $\langle\cdot,\cdot\rangle$ and ∇ when $\lambda = 0$). Each ∇^λ is torsion free and preserves the metric inner product. Using these two properties, we obtain Koszul's formula, which says for any smooth vector fields X, Y, Z on $\widetilde{M}$,

(4.10)

$$2\langle\nabla^\lambda_X Y, Z\rangle_\lambda = X\langle Y, Z\rangle_\lambda + Y\langle X, Z\rangle_\lambda - Z\langle X, Y\rangle_\lambda + \langle[X, Y], Z\rangle_\lambda - \langle[X, Z], Y\rangle_\lambda - \langle[Y, Z], X\rangle_\lambda.$$

Note that $\widetilde{g}^\lambda = e^{2\varphi^\lambda\circ\varpi}\widetilde{g}$, which means $\langle\cdot,\cdot\rangle_\lambda = e^{2\varphi^\lambda\circ\varpi}\langle\cdot,\cdot\rangle$. So, if we multiply both sides of (4.10) with $e^{-2\varphi^\lambda\circ\varpi}$ and compare it with the expression (4.10) for ∇, we obtain

$$\begin{aligned}
2\langle\nabla^\lambda_X Y, Z\rangle &= e^{-2\varphi^\lambda\circ\varpi}\left((D_X e^{2\varphi^\lambda\circ\varpi})\langle Y, Z\rangle + (D_Y e^{2\varphi^\lambda\circ\varpi})\langle X, Z\rangle - (D_Z e^{2\varphi^\lambda\circ\varpi})\langle X, Y\rangle\right)\\
&\quad + X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle[X, Y], Z\rangle - \langle[X, Z], Y\rangle - \langle[Y, Z], X\rangle\\
&= 2(D_X\varphi^\lambda\circ\varpi)\langle Y, Z\rangle + 2(D_Y\varphi^\lambda\circ\varpi)\langle X, Z\rangle - 2(D_Z\varphi^\lambda\circ\varpi)\langle X, Y\rangle + 2\langle\nabla_X Y, Z\rangle.
\end{aligned}$$

Since Z is arbitrary, this implies

$$\nabla^\lambda_X Y - \nabla_X Y = (D_X\varphi^\lambda\circ\varpi)Y + (D_Y\varphi^\lambda\circ\varpi)X - \langle X, Y\rangle\nabla\varphi^\lambda\circ\varpi$$

for any two smooth vector fields X, Y on $\widetilde{M}$. As a consequence, we have

$$\Gamma_X Y = (D_X\varphi\circ\varpi)Y + (D_Y\varphi\circ\varpi)X - \langle X, Y\rangle\nabla\varphi\circ\varpi.$$

In particular, $\Gamma_{\dot\gamma}\dot\gamma = 2\langle\nabla\varphi\circ\varpi, \dot\gamma\rangle\dot\gamma - \nabla\varphi\circ\varpi$ and the equation (4.1) reduces to

$$\nabla^2_{\dot\gamma}\Xi_\gamma + \mathrm{R}(\Xi_\gamma, \dot\gamma)\dot\gamma - \nabla\varphi\circ\varpi + \langle\nabla\varphi\circ\varpi, \dot\gamma\rangle\dot\gamma = 0.$$

The formula for $\left(\overline{X}_{\widetilde{g}^\lambda}\right)'_0(x,\xi)$ follows immediately by Proposition 4.5. $\square$

5. Proof of the main theorems

Let $\lambda \in (-1,1) \mapsto g^\lambda \in \Re(M)$ be a C^3 curve of C^3 conformal changes of the metric $g^0 = g$. We simply use the superscript λ ($\lambda \neq 0$) for $\overline{X}, \mathbf{m}, \widetilde{\mathbf{m}}, k_{\mathbf{v}}$, $\mathbb{P}$ to indicate that the metric used is g^λ, for instance, $\mathbf{m}^\lambda$ is the harmonic measure for the laminated Laplacian in metric g^λ. The corresponding quantities for g will appear without superscripts. Let $\lambda \in (-1,1) \mapsto \varphi^\lambda$ be such that $g^\lambda = e^{2\varphi^\lambda} g$. For each λ, we have

$$\Delta^\lambda = e^{-2\varphi^\lambda}\left(\Delta + (m-2)\nabla\varphi^\lambda\right) =: e^{-2\varphi^\lambda} L_\lambda.$$

Let $\widehat{\mathcal{L}}^\lambda := \Delta + Z^\lambda$ with $Z^\lambda = (m-2)\nabla\varphi^\lambda \circ \varpi$. Leafwisely, Z^λ is the dual of the closed form $(m-2)d\varphi^\lambda \circ \varpi$. Moreover, the pressure of the function $-\langle \overline{X}, Z^0\rangle = 0$ is positive. Therefore, there exists $\delta > 0$ such that for $|\lambda| < \delta$, the pressure of the function $-\langle \overline{X}, Z^\lambda\rangle$ is still positive, so that the results of Section 3 apply to $\widehat{\mathcal{L}}^\lambda$ for $\lambda \in (-\delta, +\delta)$. Note that $\widehat{\ell}_\lambda$ and $\widehat{h}_\lambda$ defined in Section 1 are just the linear drift and the stochastic entropy for the operator $\widehat{\mathcal{L}}^\lambda$ with respect to metric g. Let ℓ_λ and h_λ be the linear drift and entropy for (M, g^λ) as were defined in Section 1. From the results in Sections 3 and 4, the following limits considered in Section 1 exist:

$$\begin{aligned}
(d\ell_\lambda/d\lambda)|_{\lambda=0} &= \lim_{\lambda\to 0}\frac{1}{\lambda}(\ell_\lambda - \widehat{\ell}_\lambda) + \lim_{\lambda\to 0}\frac{1}{\lambda}(\widehat{\ell}_\lambda - \ell_0) =: (\mathbf{I})_\ell + (\mathbf{II})_\ell,\\
(dh_\lambda/d\lambda)|_{\lambda=0} &= \lim_{\lambda\to 0}\frac{1}{\lambda}(h_\lambda - \widehat{h}_\lambda) + \lim_{\lambda\to 0}\frac{1}{\lambda}(\widehat{h}_\lambda - h_0) =: (\mathbf{I})_h + (\mathbf{II})_h.
\end{aligned}$$

This shows the differentiability in λ at 0 of $\lambda \mapsto \ell_\lambda$ and $\lambda \mapsto h_\lambda$ (Theorem 1.1). In this section, we give more details and formulas for the derivative. Namely, we prove the following Theorem

Theorem 5.1. *Let (M,g) be a negatively curved compact connected m-dimensional Riemannian manifold and let $\lambda \in (-1,1) \mapsto g^\lambda = e^{2\varphi^\lambda} g \in \Re(M)$be a C^3 curve of C^3 conformal changes of the metric $g^0 = g$ with constant volume. Let φ be such that $g^\lambda = e^{2\lambda\varphi + O(\lambda^2)} g$. With the above notations, the following holds true.*

i) *The function $\lambda \mapsto \ell_\lambda$ is differentiable at 0 with*

$$\begin{aligned}
(\ell_\lambda)'_0 = &\int_{M_0\times\partial\widetilde{M}} \langle \varphi\circ\varpi\, \overline{X} + \int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{(x,\xi)}(0)\mathrm{K}_s(0))\, ds, \nabla \ln k_{\mathbf{v}}\rangle\, d\widetilde{\mathbf{m}}\\
&+(m-2)\int_{M_0\times\partial\widetilde{M}} \varphi\circ\varpi\, \langle \nabla u_0 + \overline{X}, \nabla \ln k_{\mathbf{v}}\rangle\, d\widetilde{\mathbf{m}},
\end{aligned} \tag{5.1}$$

where $(\mathrm{K}_s(0), \mathrm{K}'_s(0)) = (D\mathbf{\Phi}_s)^{-1}(0, \Upsilon(s))$ with $\Upsilon = -\nabla\varphi + \langle\nabla\varphi, \dot\gamma\rangle\dot\gamma$ along the $\widetilde{g}$-geodesic γ with $\dot\gamma(0) = (x,\xi)$ and u_0 is the function defined before Proposition 2.18.

ii) *The function* $\lambda \mapsto h_\lambda$ *is differentiable at* 0 *with*

$$(h_\lambda)'_0 = (m-2)\int_{SM} \varphi \circ \varpi \, \langle \nabla(u_1 + \ln k_{\mathbf{v}}), \nabla \ln k_{\mathbf{v}} \rangle \, d\mathbf{m}, \tag{5.2}$$

where u_1 *is the function defined before Proposition 2.18.*

Proof. Observe firstly that since g^λ has constant volume, $m \int \varphi \, d\mathrm{Vol} = \big(\mathrm{Vol}(M, g^\lambda)\big)'_0 = 0$ and therefore

$$\int_{SM} \varphi \circ \varpi \, d\mathbf{m} \; = \; 0. \tag{5.3}$$

We derive the formula for $(h_\lambda)'_0$ first. Let $\widehat{\mathbf{m}}^\lambda$ be the G-invariant extension to $S\widetilde{M}$ of the harmonic measure corresponding to $\widehat{\mathcal{L}}^\lambda$ with respect to metric g. Then $d\widehat{\mathbf{m}}^\lambda = e^{-2\varphi^\lambda \circ \varpi} d\widetilde{\mathbf{m}}^\lambda$, where φ^λ also denotes its G-invariant extension to $\widetilde{M}$. Moreover, since there is only a time change between the leafwise diffusion processes with infinitesimal operators $\widehat{\mathcal{L}}^\lambda$ and Δ^λ, the leafwise Martin kernel functions of the two operators are the same. (Indeed, because $\widehat{\mathcal{L}}^\lambda$ only differs from Δ^λ by multiplication by a positive function, the leafwise positive harmonic functions of the two generators are the same. In particular, the minimal leafwise positive harmonic functions normalized at $x = \varpi(\mathbf{v})$ are the same for $\widehat{\mathcal{L}}^\lambda$ and Δ^λ. It is known ([**Anc**, Theorem 3]) that the leafwise Martin kernel functions $k^\lambda_{\mathbf{v}}(\cdot, \xi)$ of $\widehat{\mathcal{L}}^\lambda$ (or Δ^λ) can be characterized as minimal leafwise positive $\widehat{\mathcal{L}}^\lambda$ (or Δ^λ)-harmonic functions normalized at x such that $k^\lambda_{\mathbf{v}}(y, \xi)$ goes to zero when y tends to a point in the boundary different from ξ. Thus, the two Martin kernel functions coincide.) Using Proposition 2.16, we obtain

$$\widehat{h}_\lambda = \int \|\nabla^0 \ln k^\lambda_{\mathbf{v}}(x,\xi)\|_0^2 \; d\widehat{\mathbf{m}}^\lambda = \int e^{-2\varphi^\lambda \circ \varpi} \|\nabla \ln k^\lambda_{\mathbf{v}}(x,\xi)\|^2 \; d\widetilde{\mathbf{m}}^\lambda, \tag{5.4}$$

whereas here, and hereafter, the integrals with respect to $\widehat{\mathbf{m}}^\lambda$ and $\widetilde{\mathbf{m}}^\lambda$ are always taken on $M_0 \times \partial\widetilde{M}$ and we will omit the subscript of $\int_{M_0 \times \partial \widetilde{M}}$ whenever there is no ambiguity. As before, $k^\lambda_{\mathbf{v}}(\cdot, \eta)$ should be understood as a function on $W^s(\mathbf{v})$ for all η. In particular, for $\eta = \xi$. Then its gradient (for the lifted metric from $\widetilde{M}$ to $W^s(\mathbf{v})$) is a tangent vector to $W^s(\mathbf{v})$. We also know $k^\lambda_{\mathbf{v}}(y, \eta) = k^\lambda_\eta(y)$, where k^λ_η is the Martin kernel function on $\widetilde{M}$ for the $\widetilde{g}^\lambda$-Laplacian. Of our special interest is $k^\lambda_{\mathbf{v}}(\cdot, \xi)$, which we will abbreviate as $k^\lambda_{\mathbf{v}}$ in the following context.

For $(h_\lambda)'_0$, we have

$$(h_\lambda)'_0 = \lim_{\lambda \to 0} \frac{1}{\lambda}(h_\lambda - \widehat{h}_\lambda) + \lim_{\lambda \to 0} \frac{1}{\lambda}(\widehat{h}_\lambda - h_0) =: (\mathbf{I})_h + (\mathbf{II})_h,$$

if both limits exist. It is easy to see $(\mathbf{I})_h = 0$ since by Proposition 2.16 and (5.4),

$$\begin{aligned}(\mathbf{I})_h &= \lim_{\lambda\to 0}\frac{1}{\lambda}\left(\int \|\nabla^\lambda \ln k_{\mathbf{v}}^\lambda(x,\xi)\|_\lambda^2 \ d\widetilde{\mathbf{m}}^\lambda - \int \|\nabla^0 \ln k_{\mathbf{v}}^\lambda(x,\xi)\|_0^2 \ d\widehat{\mathbf{m}}^\lambda\right)\\ &= \lim_{\lambda\to 0}\int \frac{1}{\lambda}(e^{-2\varphi^\lambda\circ\varpi} - e^{-2\varphi^\lambda\circ\varpi})\|\nabla \ln k_{\mathbf{v}}^\lambda(x,\xi)\|^2 \ d\widetilde{\mathbf{m}}^\lambda,\end{aligned}$$

where we use

$$\nabla^\lambda \ln k_{\mathbf{v}}^\lambda(x,\xi) = e^{-2\varphi^\lambda\circ\varpi}\nabla \ln k_{\mathbf{v}}^\lambda(x,\xi) \text{ and } \|\nabla^\lambda \ln k_{\mathbf{v}}^\lambda(x,\xi)\|_\lambda^2 = e^{-2\varphi^\lambda\circ\varpi}\|\nabla \ln k_{\mathbf{v}}^\lambda(x,\xi)\|^2.$$

Thus,

$$(\mathbf{I})_h \ = \ 0. \tag{5.5}$$

For $(\mathbf{II})_h$, we have by Theorem 3.9 that it equals to $\lim_{t\to+\infty}(1/t)\mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{h,t}\mathbf{M}_t)$. Recall that $\mathbf{x}_t$ belongs to $W^s(\mathbf{x}_0)$. The process

$$\widetilde{\mathbf{Z}}_t^1 \ = \ f_1(\mathbf{x}_t) - f_1(\mathbf{x}_0) - \int_0^t (\Delta f_1)(\mathbf{x}_s)\ ds, \tag{5.6}$$

where $f_1 = -\ln k_{\mathbf{v}} - u_1$ and $\mathbf{v} = \mathbf{x}_0$ and the function u_1 is such that

$$\Delta u_1 = \|\nabla \ln k_{\mathbf{v}}\|^2 - h_0 \tag{5.7}$$

is a martingale with increasing process $2\|\nabla \ln k_{\mathbf{v}} + \nabla u_1\|^2(\mathbf{x}_t)\ dt$. It is true by Proposition 3.2 that

$$\lim_{t\to+\infty}\frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{h,t}\mathbf{M}_t) = \lim_{t\to+\infty}\frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}}(\widetilde{\mathbf{Z}}_t^1\mathbf{M}_t),$$

where

$$\mathbf{M}_t = \frac{1}{2}\int_0^t \langle (Z^\lambda)_0'(\mathbf{x}_s), \mathbf{w}_s dB_s\rangle_{\mathbf{x}_s}.$$

Note that $(Z^\lambda)_0'$, the G-invariant extension of $(m-2)\nabla\varphi\circ\varpi$, is a gradient field. So, if we write $\psi = \frac{1}{2}(m-2)\varphi\circ\varpi$, we have by Ito's formula that

$$\mathbf{M}_t = \psi(\mathbf{x}_t) - \psi(\mathbf{x}_0) - \int_0^t (\Delta\psi)(\mathbf{x}_s)\ ds \tag{5.8}$$

is a martingale with increasing process $2\|\nabla\psi\|^2$. Using (5.6), (5.8) and a straightforward computation using integration by parts formula for $(a\widetilde{\mathbf{Z}}_t^1 + b\mathbf{M}_t)^2$, $a, b = 0$ or 1, we obtain

$$\widetilde{\mathbf{Z}}_t^1\mathbf{M}_t = 2\int_0^t \langle \nabla f_1, \nabla\psi\rangle(\mathbf{x}_s)\ ds$$

and hence

$$\lim_{t\to+\infty}\frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}}(\widetilde{\mathbf{Z}}_t^1\mathbf{M}_t) = 2\int \langle \nabla f_1, \nabla\psi\rangle \ \ d\widetilde{\mathbf{m}} = -2\int \langle \nabla \ln k_{\mathbf{v}}, \nabla\psi\rangle \ d\widetilde{\mathbf{m}} - 2\int \langle \nabla u_1, \nabla\psi\rangle \ d\widetilde{\mathbf{m}}.$$

Here,

$$-2\int \langle \nabla \ln k_{\mathbf{v}}, \nabla\psi\rangle \ d\widetilde{\mathbf{m}} = 2\int_{SM} \mathrm{Div}(\nabla\psi)\ d\mathbf{m} = (m-2)\int_{SM} \Delta(\varphi\circ\varpi)\ d\mathbf{m} = 0,$$

where the first equality is the integration by parts formula and $\mathbf{m}$ is identified with the restriction of $\widetilde{\mathbf{m}}$ to $M_0 \times \partial\widetilde{M}$, and the last one holds because $\mathbf{m}$ is Δ-harmonic. We finally obtain

$$(h_\lambda)'_0 = -(m-2)\int_{SM} \langle \nabla u_1, \nabla \varphi \circ \varpi \rangle \ d\mathbf{m}.$$

Observe that:

$$\begin{aligned} 2\langle \nabla u_1, \nabla\varphi\circ\varpi\rangle &= \Delta(u_1\varphi\circ\varpi) - \Delta(u_1)\varphi\circ\varpi - u_1\Delta\varphi\circ\varpi \\ &= \Delta(u_1\varphi\circ\varpi) - \varphi\circ\varpi\|\nabla \ln k_{\mathbf{v}}\|^2 + h_0\varphi\circ\varpi - u_1\Delta\varphi\circ\varpi, \end{aligned} \tag{5.9}$$

where we use the defining property (5.7) of u_1. When we take the integral of (5.9) with respect to $\mathbf{m}$, the first term vanishes because $\mathbf{m}$ is Δ harmonic, the second term gives $-\int \varphi\circ\varpi\|\nabla \ln k_{\mathbf{v}}\|^2 \ d\mathbf{m}$, the third term vanishes by (5.3). Finally for the last term, by using the integration by parts formula:

$$\int_{SM} u\Delta v \ d\mathbf{m} = \int_{SM} v\Delta u \ d\mathbf{m} + 2\int_{SM} v\langle \nabla u, \nabla \ln k_{\mathbf{v}}\rangle \ d\mathbf{m}, \tag{5.10}$$

we have

$$\begin{aligned} \int_{SM} u_1\Delta\varphi\circ\varpi \ d\mathbf{m} &= \int_{SM} \varphi\circ\varpi\,(\Delta u_1 + 2\langle\nabla u_1, \nabla\ln k_{\mathbf{v}}\rangle) \ d\mathbf{m} \\ &= \int_{SM} \varphi\circ\varpi\left(\|\nabla \ln k_{\mathbf{v}}\|^2 + 2\langle\nabla u_1, \nabla\ln k_{\mathbf{v}}\rangle\right) \ d\mathbf{m}. \end{aligned}$$

Next, we derive the formula for $(\ell_\lambda)'_0$. Clearly,

$$(\ell_\lambda)'_0 = \lim_{\lambda\to 0}\frac{1}{\lambda}(\ell_\lambda - \widehat{\ell}_\lambda) + \lim_{\lambda\to 0}\frac{1}{\lambda}(\widehat{\ell}_\lambda - \ell_0) =: (\mathbf{I})_\ell + (\mathbf{II})_\ell,$$

if both limits exist. Here the $\widehat{\ell}_\lambda$ defined in the introduction is just the linear drift for the operator $\widehat{\mathcal{L}}^\lambda$ with respect to metric g. The $(\mathbf{II})_\ell$ term can be analyzed similarly as above for $(\mathbf{II})_h$. Indeed, by Theorem 3.3, $(\mathbf{II})_\ell = \lim_{\mathrm{t}\to+\infty}(1/\mathrm{t})\mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{\ell,\mathrm{t}}\mathbf{M}_{\mathrm{t}})$. The process

$$\widetilde{\mathbf{Z}}^0_t \ = \ f_0(\mathbf{x}_t) - f_0(\mathbf{x}_0) - \int_0^t (\Delta f_0)(\mathbf{x}_s) \ ds, \tag{5.11}$$

where $f_0 = b_{\mathbf{v}} - u_0$ and the function u_0 is such that

$$\Delta u_0 = -\mathrm{Div}(\overline{X}) - \ell_0 \tag{5.12}$$

is a martingale with increasing process $2\|\overline{X} + \nabla u_0\|^2(\mathbf{x}_t) \ dt$. It is true by Proposition 3.1 that

$$\lim_{t\to+\infty}\frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}}(\mathbf{Z}_{\ell,t}\mathbf{M}_t) = \lim_{t\to+\infty}\frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}}(\widetilde{\mathbf{Z}}^0_t\mathbf{M}_t),$$

where $\mathbf{M}_t$, by (5.8), is a martingale with increasing process $2\|\nabla\psi\|^2$. So using (5.8), (5.11) and a straightforward computation using integration by parts formula for $(a\widetilde{\mathbf{Z}}^0_t + b\mathbf{M}_t)^2$,

$a, b = 0$ or 1, we obtain

$$\widetilde{\mathbf{Z}}_t^0 \mathbf{M}_t = 2\int_0^t \langle \nabla f_0, \nabla \psi \rangle (\mathbf{x}_s)\ ds$$

and hence (recall that $\nabla b_{\mathbf{v}} = -\overline{X}(\mathbf{v})$, see (2.13))

$$\begin{aligned}\lim_{t\to+\infty} \frac{1}{t}\mathbb{E}_{\overline{\mathbb{Q}}}(\widetilde{\mathbf{Z}}_t^0 \mathbf{M}_t) &= 2\int \langle \nabla f_0, \nabla \psi \rangle\ \ d\widetilde{\mathbf{m}} \\ &= -(m-2)\left(\int \langle \overline{X}, \nabla(\varphi\circ\varpi)\rangle\ d\widetilde{\mathbf{m}} + \int \langle \nabla u_0, \nabla(\varphi\circ\varpi)\rangle\ d\widetilde{\mathbf{m}} \right).\end{aligned}$$

Using the formula $\mathrm{Div}(\varphi\circ\varpi\ \overline{X}) = \varphi\circ\varpi\ \mathrm{Div}\overline{X} + \langle \nabla(\varphi\circ\varpi), \overline{X}\rangle$, we obtain

$$\begin{aligned}\int \langle \overline{X}, \nabla(\varphi\circ\varpi)\rangle\ d\widetilde{\mathbf{m}} &= \int \left(\mathrm{Div}(\varphi\circ\varpi\ \overline{X}) - \varphi\circ\varpi\ \mathrm{Div}\overline{X}\right)\ d\widetilde{\mathbf{m}} \\ &= -\int \varphi\circ\varpi \left(\langle \overline{X}, \nabla \ln k_{\mathbf{v}}\rangle + \mathrm{Div}\overline{X}\right)\ d\widetilde{\mathbf{m}},\end{aligned}$$

where we used the foliated integration by parts formula $\int \mathrm{Div}Y\ d\widetilde{\mathbf{m}} = -\int \langle Y, \nabla \ln k_{\mathbf{v}}\rangle\ d\widetilde{\mathbf{m}}$. Observe that:

$$\begin{aligned}2\langle \nabla u_0, \nabla(\varphi\circ\varpi)\rangle &= \Delta(u_0\ \varphi\circ\varpi) - \Delta(u_0)\varphi\circ\varpi - u_0\Delta(\varphi\circ\varpi) \\ &= \Delta(u_0\ \varphi\circ\varpi) + \varphi\circ\varpi\ \mathrm{Div}(\overline{X}) + \ell_0\varphi\circ\varpi - u_0\Delta(\varphi\circ\varpi),\end{aligned}$$

where we use the defining property (5.12) of u_0. When we report in the integration $2\int\langle \nabla u_0, \nabla(\varphi\circ\varpi)\rangle\ d\widetilde{\mathbf{m}}$, the first term vanishes because $\mathbf{m}$ is Δ harmonic, the second term is $-\int \varphi\circ\varpi\Delta u_0\ d\widetilde{\mathbf{m}}$ by (5.12) and the third term vanishes by (5.3). Again, using the integration by parts formula (5.10) for $\int u_0\Delta(\varphi\circ\varpi)\ d\widetilde{\mathbf{m}}$, we have

$$\int \langle \nabla u_0, \nabla(\varphi\circ\varpi)\rangle\ d\widetilde{\mathbf{m}} = -\int \varphi\circ\varpi(\Delta u_0 + \langle \nabla u_0, \nabla \ln k\rangle)\ d\widetilde{\mathbf{m}}$$

Finally, we obtain

$$\begin{aligned}(\mathbf{II})_\ell &= (m-2)\int \varphi\circ\varpi \left(\Delta u_0 + \mathrm{Div}\overline{X} + \langle \nabla u_0 + \overline{X}, \nabla \ln k_{\mathbf{v}}\rangle\right)\ d\widetilde{\mathbf{m}} \\ &= (m-2)\int \varphi\circ\varpi \langle \nabla u_0 + \overline{X}, \nabla \ln k_{\mathbf{v}}\rangle\ d\widetilde{\mathbf{m}},\end{aligned}$$

where the last equality holds by using (5.12) and (5.3).

For $(\mathbf{I})_\ell$, we first observe the convergence of Martin kernels and harmonic measures. For any $(x,\xi) =: \mathbf{v} \in \widetilde{M}\times\partial\widetilde{M}$, the Martin kernel function $k_{\mathbf{v}}^\lambda(y,\xi)$ converges to $k_{\mathbf{v}}(y,\xi)$ pointwisely as λ goes to zero. For small λ and fixed x, the function $\xi \mapsto \nabla \ln k_{x,\xi}^\lambda$ is Hölder continuous on $\partial\widetilde{M}$ for some uniform exponent ([**H1**]). As a consequence, we have the convergence of $\nabla \ln k_{\mathbf{v}}^\lambda$ (and hence $\nabla^\lambda \ln k_{\mathbf{v}}^\lambda$) to $\nabla \ln k_{\mathbf{v}}$ when λ tends to zero. By uniqueness, the harmonic measure $\widetilde{\mathbf{m}}^\lambda$ converges weakly to $\widetilde{\mathbf{m}}$ $(\lambda \to 0)$ as well. By Proposition 2.9,

$$\ell_\lambda = \int \langle \overline{X}^\lambda, \nabla^\lambda \ln k_{\mathbf{v}}^\lambda\rangle_\lambda\ d\widetilde{\mathbf{m}}^\lambda = \int \langle \overline{X}^\lambda, \nabla \ln k_{\mathbf{v}}^\lambda\rangle\ d\widetilde{\mathbf{m}}^\lambda.$$

Thus,

$$\begin{aligned}(\mathbf{I})_\ell &= \lim_{\lambda\to 0}\frac{1}{\lambda}\int\langle(\overline{X}^\lambda-\overline{X}^0),\nabla\ln k_{\mathbf{v}}\rangle\, d\widetilde{\mathbf{m}}+\lim_{\lambda\to 0}\frac{1}{\lambda}\left(\int\langle\overline{X},\nabla\ln k_{\mathbf{v}}^\lambda\rangle\, d\widetilde{\mathbf{m}}^\lambda-\widehat{\ell}_\lambda\right)\\ &=: (\mathbf{III})_\ell+(\mathbf{IV})_\ell\end{aligned}$$

if $(\mathbf{III})_\ell$ and $(\mathbf{IV})_\ell$ exist. The quantity $(\mathbf{III})_\ell$, by Corollary 4.6, is

$$\int\langle-\varphi\circ\varpi\ \overline{X}+\int_0^{+\infty}(\mathrm{K}_s'(0)-S_{\mathbf{v}}'(0)\mathrm{K}_s(0))\ ds,\nabla\ln k_{\mathbf{v}}\rangle\ d\widetilde{\mathbf{m}}.$$

By Proposition 2.9,

$$\widehat{\ell}_\lambda=-\int\left(\mathrm{Div}\overline{X}+\langle Z^\lambda,\overline{X}\rangle\right)\ d\widehat{\mathbf{m}}^\lambda.$$

For $(\mathbf{IV})_\ell$, let us first calculate $\int\mathrm{Div}\overline{X}\ d\widehat{\mathbf{m}}$. We have

$$\begin{aligned}\int\mathrm{Div}\overline{X}\ d\widehat{\mathbf{m}}^\lambda &= \int e^{-2\varphi^\lambda\circ\varpi}\mathrm{Div}\overline{X}\ d\widetilde{\mathbf{m}}^\lambda\\ &= \int e^{-2\varphi^\lambda\circ\varpi}\mathrm{Div}^\lambda\overline{X}\ d\widetilde{\mathbf{m}}^\lambda+\int\left(\mathrm{Div}\overline{X}-\mathrm{Div}^\lambda\overline{X}\right)\ d\widehat{\mathbf{m}}^\lambda\\ &= \int e^{-2\varphi^\lambda\circ\varpi}\mathrm{Div}^\lambda\overline{X}\ d\widetilde{\mathbf{m}}^\lambda-m\int\langle\nabla(\varphi^\lambda\circ\varpi),\overline{X}\rangle\ d\widehat{\mathbf{m}}^\lambda,\end{aligned}$$

where the last equality holds since $(\mathrm{Div}^\lambda-\mathrm{Div})(\cdot)=m\langle\nabla(\varphi^\lambda\circ\varpi,\cdot\rangle$ for $g^\lambda=e^{2\varphi^\lambda}g$. Note that

$$\mathrm{Div}^\lambda(e^{-2\varphi^\lambda\circ\varpi}\overline{X})=e^{-2\varphi^\lambda\circ\varpi}\mathrm{Div}^\lambda\overline{X}-2e^{-2\varphi^\lambda\circ\varpi}\langle\nabla^\lambda(\varphi^\lambda\circ\varpi),\overline{X}\rangle_\lambda.$$

So we have

$$\begin{aligned}\int\mathrm{Div}\overline{X}\ d\widehat{\mathbf{m}}^\lambda &= \int\mathrm{Div}^\lambda(e^{-2\varphi^\lambda\circ\varpi}\overline{X})\ d\widetilde{\mathbf{m}}^\lambda+\int 2e^{-2\varphi^\lambda\circ\varpi}\langle\nabla^\lambda\varphi^\lambda\circ\varpi,\overline{X}\rangle_\lambda\ d\widetilde{\mathbf{m}}^\lambda\\ &\qquad -m\int\langle\nabla(\varphi^\lambda\circ\varpi),\overline{X}\rangle\ d\widehat{\mathbf{m}}^\lambda\\ &= -\int\langle\overline{X},\nabla^\lambda\ln k_{\mathbf{v}}^\lambda\rangle_\lambda\ d\widehat{\mathbf{m}}^\lambda-(m-2)\int\langle\nabla(\varphi^\lambda\circ\varpi),\overline{X}\rangle\ d\widehat{\mathbf{m}}^\lambda\\ &= -\int\langle\overline{X},\nabla\ln k_{\mathbf{v}}^\lambda\rangle\ d\widehat{\mathbf{m}}^\lambda-(m-2)\int\langle\nabla(\varphi^\lambda\circ\varpi),\overline{X}\rangle\ d\widehat{\mathbf{m}}^\lambda,\end{aligned}$$

where, for the second equality, we use the leafwise integration by parts formula $\int\mathrm{Div}^\lambda Y\ d\widetilde{\mathbf{m}}^\lambda=-\int\langle Y,\nabla^\lambda\ln k_{\mathbf{v}}^\lambda\rangle_\lambda\ d\widetilde{\mathbf{m}}^\lambda$. This gives

$$\widehat{\ell}_\lambda=\int\langle\overline{X},\nabla\ln k_{\mathbf{v}}^\lambda\rangle\ d\widehat{\mathbf{m}}^\lambda. \tag{5.13}$$

Finally, we obtain

$$(\mathbf{IV})_\ell=\lim_{\lambda\to 0}\int\frac{1}{\lambda}(e^{2\varphi^\lambda\circ\varpi}-1)\langle\overline{X},\nabla\ln k_{\mathbf{v}}^\lambda\rangle\ d\widehat{\mathbf{m}}^\lambda=2\int\varphi\circ\varpi\ \langle\overline{X},\nabla\ln k_{\mathbf{v}}\rangle\ d\widetilde{\mathbf{m}}.$$

□

Proof of Theorem 1.2. Let (M, g) be a negatively curved compact connected Riemannian manifold. Define the *volume entropy* v_g by:

$$v_g \; = \; \lim_{r\to+\infty} \frac{\ln \mathrm{Vol}(B(x,r))}{r},$$

where $B(x,r)$ is the ball of radius r in $\widetilde{M}$. we have $\ell_g \leq v_g$, $h_g \leq \; v_g^2$ (see [**LS1**] and the references within). In particular, if $\lambda \in (-1,1) \mapsto g^\lambda \in \Re(M)$ is a C^3 curve of conformal changes of the metric $g^0 = g$,

$$\ell_{g^\lambda} \; \leq \; v_{g^\lambda}, \quad h_{g^\lambda} \; \leq \; v_{g^\lambda}^2.$$

Assume (M, g^0) is locally symmetric. Then $\ell_{g^0} = v_{g^0}$ and $h_{g^0} = v_{g^0}^2$. Moreover it is known (Katok [**Ka**]) that v_0 is a global minimum of the volume entropy among metrics g which are conformal to g^0 and have the same volume and (Katok-Knieper-Pollicott-Weiss [**KKPW**]) that $\lambda \mapsto v_{g^\lambda}$ is differentiable. In particular v_{g^λ} is critical at $\lambda = 0$. Since, by Theorem 1.1, ℓ_{g^λ} and h_{g^λ} are differentiable at $\lambda = 0$, they have to be critical as well. $\square$

Remark 5.2. We can also show Theorem 1.2 using the formulas in Theorem 5.1. Indeed, the conclusion for the stochastic entropy follows from (5.2) since for a locally symmetric space, the solutions u_1 to (5.7) are constant ([**L2**]) and $\|\nabla \ln k_{\mathbf{v}}\|^2$ is also constant. The derivative is proportional to $\int \varphi \circ \varpi \; d\mathbf{m}$, which vanishes by (5.3).

We also see that the stochastic entropy depends only on the volume for surfaces ($m = 2$). For the drift ℓ, it is true that for a locally symmetric space, $\nabla \ln k_{\mathbf{v}} = -\ell \nabla b_{\mathbf{v}}$ everywhere. The solutions u_0 to (5.12) are constant for a locally symmetric space as well ([**L2**]). So (5.1) reduces to

$$(\ell_\lambda)'_0 = -\int_{M_0\times\partial\widetilde{M}} \varphi \circ \varpi \; \langle \int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0)) \; ds, \nabla \ln k_{\mathbf{v}} \rangle \; d\widetilde{\mathbf{m}},$$

which is zero because the vector $\int_0^{+\infty} (\mathrm{K}'_s(0) - S'_{\mathbf{v}}(0)\mathrm{K}_s(0)) \; ds$ is orthogonal to $\mathbf{v}$ and hence is orthogonal to $\nabla \ln k_{\mathbf{v}}$.

Acknowledgments The second author was partially supported by NSFC (No.11331007 and 11422104) and Beijing Higher Education Young Elite Teacher Project. She would also like to thank LPMA and the Department of Mathematics of the University of Notre Dame for hospitality during her stays. We are really grateful to the careful reading and the numerous remarks of the referee, which help us to improve the writing.

References

[**Anc**] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, *Ann. Math. (2)* **125** (1987) 495–536.

[**Ano1**] D. V. Anosov, *Geodesic Flow on Closed Riemannian Manifolds with Negative Curvature*, Proc. Steklov Inst. Math. **90** (1967).

[**Ano2**] D. V. Anosov, *Tangent fields of transversal foliations in U-systems*, Math. Notes Acad. Sci. USSR 2:5 (1968), 818–823.

[**Ba**] W. Ballmann, Lectures on spaces of nonpositive curvature, With an appendix by Misha Brin. *DMV Seminar*, **25**, Birkhuser Verlag, Basel, 1995.

[**Bi**] P. Billingsley, *Probability and measure*, Third edition, Wiley Series in Probability and Mathematical Statistics, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995.

[**Bo**] J. Bolton, Conditions under which a geodesic flow is Anosov, *Math. Ann.* **240** (1979), 103–113.

[**BHM**] S. Blachère, P. Haïssinsky and P. Mathieu, Asymptotic entropy and Green speed for random walks on countable groups, *Ann. Probab.* **36** (2008), 1134–1152.

[**DGM**] A. Debiard, B. Gaveau and E. Mazet, Théorèmes de comparaison en géométrie riemannienne, *Publ. Res. Inst. Math. Sci.* **12** (1976), 391–425.

[**EO**] P. Eberlein and B. O'Neill, Visibility manifolds, *Pacific Journal of Mathematics*, **46** (1973), 45–109.

[**El**] K. D. Elworthy, *Stochastic Differential Equations on Manifolds*, London Mathematical Society Lecture Note Series, 70. Cambridge University Press, Cambridge-New York, 1982.

[**Esc**] J.-H. Eschenburg, Horospheres and the stable part of the geodesic flow, *Math. Z.* **153** (1977), 237–251.

[**FF**] A. Fathi and L. Flaminio, Infinitesimal conjugacies and Weil-Petersson metric, *Ann. Inst. Fourier (Grenoble)* **43** (1993), 279–299.

[**Ga**] L. Garnett, Foliations, the ergodic theorem and Brownian motion, *J. Funct. Anal.* **51** (1983), 285–311.

[**GH**] E. Ghys and P. de la Harpe, Sur les groupes hyperboliques d'après Mikhael Gromov, *Progress in Mathematics* Vol. 83, Birkhäuser Boston Inc., Boston, MA, 1990.

[**Gre**] L. W. Green, A theorem of E. Hopf, *Michigan Math. J.* **5** (1958), 31–34.

[**Gro**] M. Gromov, Three remarks on geodesic dynamics and fundamental group, *Enseign. Math.* (2) **46** (2000), 391–402.

[**Gu**] Y. Guivarc'h, Sur la loi des grands nombres et le rayon spectral d'une marche aléatoire, *Astérisque,* **74** (1980), 47–98.

[**H1**] U. Hamenstädt, An explicite description of harmonic measure, *Math. Z.* **205** (1990), 287–299.

[**H2**] U. Hamenstädt, Harmonic measures for compact negatively curved manifolds, *Acta Math.* **178** (1997), 39–107.

[**HIH**] E. Heintze and H. C. Im Hof, Geometry of horospheres, *J. Differential Geometry* **12** (1977), 481–491.

[**HPS**] M. Hirsch, C. Pugh and M. Shub, *Invariant Manifolds,* Lecture Notes in Math. **583**, Springer, Berlin, 1977.

[**Hs**] E.-P. Hsu, *Stochastic Analysis on Manifolds*, Graduate Studies in Mathematics, 38. American Mathematical Society, Providence, RI, 2002.

[**K1**] V. A. Kaimanovich, Brownian motion and harmonic functions on covering manifolds. An entropic approach, *Soviet Math. Dokl.* **33** (1986), 812–816.

[**K2**] V. A. Kaimanovich, Invariant measures for the geodesic flow and measures at infinity on negatively curved manifolds, *Ann. Inst. Henri Poincaré, Physique Théorique* **53** (1990), 361–393.

[**Ka**] A. Katok, Four applications of conformal equivalence to geometry and dynamics, *Ergod. Theory Dynam. Systems* **8** (1988), 139–152.

[**KKPW**] A. Katok, G. Knieper, M. Pollicott and H. Weiss, Differentiability and analyticity of topological entropy for Anosov and geodesic flows, *Invent. math.* **98** (1989), 581–597.

[**Ki**] J. F. C. Kingman, The ergodic theory of subadditive stochastic processes, *J. Royal Stat. Soc.* **B30** (1968), 499–510.

[**L1**] F. Ledrappier, Ergodic properties of Brownian motion on covers of compact negatively-curved manifolds, *Bol. Soc. Bras. Mat.* **19** (1988), 115–140.

[**L2**] F. Ledrappier, Central limit theorem in negative curvature, *Ann. Probab.* **23** (1995), 1219–1233.

[**L3**] F. Ledrappier, Profil d'entropie dans le cas continu, *Astérisque* **236** (1996), 189–198.

[**LS1**] F. Ledrappier and L. Shu, Entropy rigidity of symmetric spaces without focal points, *Trans. Amer. Mat. Soc.* **366** (2014), 3805–3820.

[**LS2**] F. Ledrappier and L. Shu, in prepration.

[**LY**] P. Li and S.-T. Yau, On the parabolic kernel of the Schrödinger operator, *Acta Math.* **156** (1986), 153–201.

[**LMM**] R. de la Llave, J.-M. Marco and R. Moriyón, Canonical perturbation theory of Anosov systems and regularity results for the Livsic cohomology equation, *Ann. of Math.* (2) **123** (1986), 537–611.

[**Ma**] P. Mathieu, Differentiating the entropy of random walks on hyperbolic groups, *Ann. Prob.* **43** (2015) 166–187.

[**Moh**] O. Mohsen, *Familles de mesures au bord et bas du spectre*, Thesis, École Polytechnique, 2007.

[**Mor**] M. Morse, A fundamental class of geodesics on any closed surface of genus greater than one, *Trans. Amer. Math. Soc.* **26** (1924), 25–60.

[**Mos**] J. Moser, On Harnack's theorem for elliptic differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577–591.

[**N**] A. A. Novikov, On moment inequalities and identities for stochastic integrals, Proceedings of the Second Japan-USSR Symposium on Probability Theory (Kyoto, 1972), pp. 333–339 in *Lecture Notes in Math., Vol. 330, Springer, Berlin, 1973.*

[**PPS**] F. Paulin, M. Pollicott and B. Schapira, Equilibrium states in negative curvature, *Astérisque*, Soc. Math. France, **373** (2015).

[**RY**] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, Grundlehren der Mathematischen Wissenschaften, 293. Springer-Verlag, Berlin, 1999.

[**SFL**] M. Shub, A. Fathi and R. Langevin, *Global stability of dynamical systems*, Springer-Verlag, New York-Berlin, 1987.

[**Sa**] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, *J. Differential Geom.* **36** (1992), 417–450.

[**St**] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier (Grenoble)* **15** (1965), 189–258.

[**T**] N. S. Trudinger, Linear elliptic operators with measurable coefficients, *Ann. Scuola Norm. Sup. Pisa* (3) **27** (1973), 265–308.

[**W**] F.-Y. Wang, Sharp explicit lower bounds of heat kernels, *Ann. Probab.* **25** (1997), 1995–2006.

[**Y**] C. Yue, Brownian motion on Anosov foliations and manifolds of negative curvature, *J. Diff. Geom.* **41** (1995), 159–183.

François Ledrappier, Department of Mathematics, University of Notre Dame, IN 46556-4618, USA and LPMA, Boîte Courrier 188, 4, Place Jussieu, 75252 PARIS cedex 05, France

E-mail address: `fledrapp@nd.edu`

Lin Shu, LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

E-mail address: `lshu@math.pku.edu.cn`